Published by Institute of Physics Publishing for SISSA/ISAS
Received: *December 3, 2003*
Accepted: *December 15, 2003*

Flavoring the gravity dual of $\mathcal{N}=1$ Yang-Mills with probes

Carlos Núñez

Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.
E-mail: nunez@lns.mit.edu

Ángel Paredes and Alfonso V. Ramallo

Departamento de Física de Partículas, Universidad de Santiago de Compostela
E-15782 Santiago de Compostela, Spain
E-mail: angel@fpaxp1.usc.es, alfonso@fpaxp1.usc.es

Abstract: We study two related problems in the context of a supergravity dual to $\mathcal{N}=1$ SYM. One of the problems is finding kappa symmetric D5-brane probes in this particular background. The other is the use of these probes to add flavors to the gauge theory. We find a rich and mathematically appealing structure of the supersymmetric embeddings of a D5-brane probe in this background. Besides, we compute the mass spectrum of the low energy excitations of $\mathcal{N}=1$ SQCD (mesons) and match our results with some field theory aspects known from the study of supersymmetric gauge theories with a small number of flavors.

Keywords: D-branes, Brane Dynamics in Gauge Theories, AdS-CFT and dS-CFT Correspondence, Supersymmetry and Duality.

Contents

1. Introduction **2**
  1.1 Reader's guide 4

2. The supergravity dual of $\mathcal{N} = 1$ Yang-Mills **5**

3. Kappa symmetry **7**

4. Branes wrapped at fixed distance **9**

5. Worldvolume solitons (abelian case) **12**
  5.1 n-Winding solitons 14
  5.2 (n,m)-Winding solitons 17
  5.3 Spiral solitons 18

6. Worldvolume solitons (nonabelian case) **19**
  6.1 Non abelian unit-winding solutions 20
  6.2 Non abelian zero-winding solutions 23
  6.3 Cylinder solutions 25

7. Quadratic fluctuations around the unit-winding embedding **26**
  7.1 Scalar mesons 27
  7.2 Vector mesons 35
  7.3 Gauge theory interpretation 35

8. Summary and conclusions **37**

A. Supersymmetry of the $\mathcal{N} = 1$ supergravity duals **38**
  A.1 Fivebranes wrapped on a two-cycle 39
  A.2 Fivebranes wrapped on a three-cycle 42

B. Asymptotic behaviour of the fluctuations **45**
  B.1 Non abelian unit-winding embedding 45
  B.2 n-Winding embeddings in the abelian background 47

1. Introduction

The gauge/string correspondence, an old proposal due to 't Hooft [1], is now well understood in the context of maximally supersymmetric super Yang-Mills (SYM) theories. Indeed, the so-called AdS/CFT correspondence is a conjectured equivalence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory [2]. In the large 't Hooft coupling limit, the $\mathcal{N} = 4$ SYM theory is dual to the type IIB supergravity background corresponding to the near-horizon geometry of a stack of parallel D3-branes, whose metric is precisely that of the $AdS_5 \times S^5$ space. There are nowadays a lot of non-trivial tests of this duality (for a review see [3]).

The extension of the gauge/string correspondence to theories with less supersymmetries is obviously of great interest. A possible way to obtain supergravity duals of SYM theories with reduced supersymmetry is to consider branes wrapping supersymmetric cycles of Calabi-Yau manifolds [4]. In order to preserve some supersymmetry the normal bundle of the cycle within the Calabi-Yau space has to be twisted [5]. Gauged supergravities in lower dimensions provide the most natural framework to implement this twisting. In these theories the gauge field can be used to fiber the cycle in which the brane is wrapped in such a way that some supersymmetries are preserved.

In this paper we will restrict ourselves to the case of the supergravity dual of $\mathcal{N} = 1$ SYM. This background, which corresponds to a fivebrane wrapping a two-cycle, was obtained in ref. [6] from the solution found in ref. [7] representing non-abelian magnetic monopoles in four dimensions. The geometry of this background is smooth and leads to confinement and chiral symmetry breaking. Actually, if only the abelian part of the vector field of seven dimensional gauged supergravity is excited, one obtains a geometry which is singular at the origin and coincides with the smooth one at large distances, i.e. in the UV. Therefore, the singularity at the origin is resolved by making the gauge field non-abelian, in complete analogy to what happens with the resolution of the Dirac string by the 't Hooft-Polyakov monopole. Moreover, as argued in ref. [8], the same mechanism that de-singularizes the supergravity solution also gives rise to gaugino condensation. Based on this observation, the NSVZ beta function can be reproduced at leading order [9, 10, 11]. Other aspects of this supergravity dual have been studied in ref. [12](for a review see [13]).

Most of the analysis carried out with the background of [6] do not incorporate quarks in the fundamental representation which, in a string theory setup, correspond to open strings. In order to introduce an open string sector in a supergravity dual it is quite natural to add D-brane probes and see whether one can extract some information about the quark dynamics. As usual, if the number of brane probes is much smaller than those of the background, one can assume that there is no backreaction of the probe in the bulk geometry. In this paper we follow this approach and we will probe with D5-branes the supergravity dual of $\mathcal{N} = 1$ SYM. The main technique to determine the supersymmetric brane probe configurations is kappa symmetry [14], which tells us that, if ϵ is a Killing spinor of the background, only those embeddings for which a certain matrix Γ_κ satisfies

$$\Gamma_\kappa \, \epsilon = \epsilon \,, \tag{1.1}$$

preserve the supersymmetry of the background [15]. The matrix Γ_κ depends on the metric induced on the worldvolume of the brane. Therefore, if the Killing spinors ϵ are known, we can regard (1.1) as an equation for the embedding of the brane.

The starting point in our program will be the determination of the Killing spinors of the background. It turns out that a simple expression for these spinors can be obtained if one considers a frame inspired by the uplifting of the metric from gauged seven dimensional supergravity. The realization of the topological twist in this case is similar to the one introduced in ref. [16], which generalizes that of ref. [17], to obtain manifolds of G_2 holonomy and the deformed and resolved conifold from gauged eight-dimensional supergravity. The resulting Killing spinors are characterized by a series of projections and all we have to do is to find those configurations for which the kappa symmetry condition (1.1) follows from the projections satisfied by the Killing spinors.

The probes we are going to consider are D5-branes wrapped on a two-dimensional submanifold. We will be able to find some differential equations for the embedding which are, in general, quite complicated to solve. The first obvious configuration one should look at is that of a fivebrane wrapped at a fixed distance from the origin. In this case the equations simplify drastically and we will be able to prove a no-go theorem which states that, unless we place the brane at an infinite distance from the origin, the probe breaks supersymmetry. This result is consistent with the fact that these $\mathcal{N}=1$ theories do not have a moduli space. In this analysis we will make contact with the two-cycle considered in ref. [10] and show that it preserves supersymmetry at an asymptotically large distance from the origin.

Guided by the negative result obtained when trying to wrap the D5-brane at constant distance, we will allow this distance to vary within the two-submanifold of the embedding. To simplify the equations that determine the embeddings, we first consider the singular version of the background, in which the vector field of the seven dimensional gauged supergravity is abelian. This geometry coincides with the non-singular one, in which the vector field is non-abelian, at large distances from the origin. By choosing an appropriate set of variables we will be able to write the differential equations for the embedding as two pairs of Cauchy-Riemann equations which are straightforward to integrate in general. Among all possible solutions we will concentrate on some of them characterized by integers, which can be interpreted as winding numbers. Generically these solutions have spikes, in which the probe is at infinite distance from the origin and, thus, they correspond to fivebranes wrapping a non compact submanifold. Moreover, these configurations are worldvolume solitons and we will verify that they saturate an energy bound [18].

With the insight gained by the analysis of the worldvolume solitons in the abelian background we will consider the equations for the embeddings in the non-abelian background. In principle any solution for the smooth geometry must coincide in the UV with one of the configurations found for the singular metric. This observation will allow us to formulate an ansatz to solve the complicated equations arising from kappa symmetry. Actually, in some cases, we will be able to find analytical solutions for the embeddings, which behave as those found for the singular metric at large distance from the origin and also saturate an energy bound, which ensures their stability.

One of our motivations to study brane probes is to use these results to explore the quark sector of the gauge/gravity duality. Actually, it was proposed in refs. [19, 20] that one

JHEP12(2003)024

can add flavor to this correspondence by considering spacetime filling branes and looking at their fluctuations. In ref. [21] this program has been made explicit for the $AdS_5 \times S^5$ geometry of a stack of D3-branes and a D7-brane probe. When the D3-branes of the background and the D7-brane of the probe are separated, the fundamental matter arising from the strings stretched between them becomes massive and a discrete spectrum of mesons for an $\mathcal{N} = 2$ SYM with a matter hypermultiplet can be obtained analytically from the fluctuations of a D7-brane probe. In ref. [22] a similar analysis was performed for the $\mathcal{N} = 1$ Klebanov-Strassler background [23], while in refs. [24, 25] the meson spectrum for some non-supersymmetric backgrounds was found (for recent related work see refs. [26, 27]).

It was suggested in ref. [28] that one possible way to add flavor to the $\mathcal{N} = 1$ SYM background is by considering supersymmetric embeddings of D5-branes which wrap a two-dimensional submanifold and are spacetime filling. Some of the configurations we will find in our kappa symmetry analysis have the right ingredients to be used as flavor branes. They are supersymmetric by construction, extend infinitely and have some parameter which determines the minimal distance between the brane probe and the origin. This distance should be interpreted as the mass scale of the quarks. Moreover, these brane probes capture geometrically the pattern of R-symmetry breaking of SQCD with few flavors [29]. Consequently, we will study the quadratic fluctuations around the static probe configurations found by integrating the kappa symmetry equations. We will verify that these fluctuations decay exponentially at large distances. However, we will not be able to define a normalizability condition which could give rise to a discrete spectrum. The reason for this is the exponential blow up of the dilaton at large distances. Actually, this same difficulty was found in ref. [30] in the study of the glueball spectrum for this background. As proposed in ref. [30], we shall introduce a cut-off and impose boundary conditions which ensure that the fluctuation takes place in a region in which the supergravity approximation remains valid. The resulting spectrum is discrete and, by using numerical methods, we will be able to determine its form.

The organization of this paper is the following. In section 2 we introduce the supergravity dual of $\mathcal{N} = 1$ SYM. The Killing spinors for this background are obtained in appendix A, where we also obtain those corresponding to the background of refs. [31, 32, 33], which represents D5-branes wrapped on a three-cycle. In section 3 we obtain the kappa symmetry equations which determine the supersymmetric embeddings. In section 4 we obtain the no-go theorem for branes wrapped at fixed distance. In section 5 the kappa symmetry equations for the abelian background are integrated in general and some of the particular solutions are studied in detail. Section 6 deals with the integration of the equations for the supersymmetric embeddings in the full non-abelian background. The spectrum of the quadratic fluctuations is analysed in section 7. The asymptotic form of these fluctuations is obtained in appendix B. Finally, in section 8 we summarize our results, draw some conclusions and discuss some lines of future work.

1.1 Reader's guide

Given that this is a long paper, we feel it would be useful to include here a "roadmap" to help the reader to quickly find the results of his/her particular interest. Those readers

JHEP12(2003)024

interested in the supersymmetry preserved by the background and in the application of kappa symmetry to find compact and non compact embeddings in the geometry dual to $\mathcal{N}=1$ SYM, should pay special attention to sections 2–6 and appendix A. Readers more interested in the gravity version of the addition of flavors to $\mathcal{N}=1$ SYM should take for granted section 3 and look at the solutions exhibited in eqs. (5.19), (5.22) and (6.15), which are what we called "abelian and non-abelian unit-winding solutions". Then, they should go straight to section 7 and take into account the results of appendix B.

2. The supergravity dual of $\mathcal{N}=1$ Yang-Mills

The supergravity solution we will be dealing with corresponds to a stack of N D5-branes wrapped on a two-cycle. It can be obtained [6, 7] by considering seven dimensional gauged supergravity, which is a consistent truncation of ten dimensional supergravity on a three-sphere. To get this background one starts with an ansatz for the seven dimensional metric which has a term corresponding to the metric of a two-sphere and looks for a supersymmetric solution of the equations of motion. After uplifting to ten dimensions one ends up with a solution of type IIB supergravity which preserves four supersymmetries. The ten dimensional metric in Einstein frame is:

$$ds_{10}^2 = e^{\frac{\phi}{2}}\left[dx_{1,3}^2 + e^{2h}(d\theta^2 + \sin^2\theta d\varphi^2) + dr^2 + \frac{1}{4}(w^i - A^i)^2\right], \tag{2.1}$$

where ϕ is the dilaton, the unwrapped coordinates x^μ have been rescaled and all distances are measured in units of $Ng_s\alpha'$. The angles $\theta \in [0,\pi]$ and $\varphi \in [0, 2\pi)$ parametrize the two-sphere of gauged seven dimensional supergravity. This sphere is fibered in the ten dimensional metric by the one-forms A^i ($i = 1, 2, 3$), which are the components of the non-abelian gauge vector field of the seven dimensional supergravity. Their expression can be written in terms of a function $a(r)$ and the angles (θ, φ) as follows:

$$A^1 = -a(r)d\theta\,, \qquad A^2 = a(r)\sin\theta d\varphi\,, \qquad A^3 = -\cos\theta d\varphi\,. \tag{2.2}$$

The w^i's appearing in eq. (2.1) are the $su(2)$ left-invariant one-forms, satisfying $dw^i = -\frac{1}{2}\,\epsilon_{ijk}\, w^j \wedge w^k$, which parametrize the compactification three-sphere and can be represented in terms of three angles $\tilde{\varphi}$, $\tilde{\theta}$ and ψ:

$$\begin{aligned} w^1 &= \cos\psi d\tilde{\theta} + \sin\psi \sin\tilde{\theta} d\tilde{\varphi}\ , \\ w^2 &= -\sin\psi d\tilde{\theta} + \cos\psi\sin\tilde{\theta} d\tilde{\varphi}\ , \\ w^3 &= d\psi + \cos\tilde{\theta} d\tilde{\varphi}\,. \end{aligned} \tag{2.3}$$

The three angles $\tilde{\varphi}$, $\tilde{\theta}$ and ψ take values in the rank $0 \le \tilde{\varphi} < 2\pi$, $0 \le \tilde{\theta} \le \pi$ and $0 \le \psi < 4\pi$. For a metric ansatz such as the one written in (2.1) one obtains a supersymmetric solution when the functions $a(r)$, $h(r)$ and the dilaton ϕ are:

$$\begin{aligned} a(r) &= \frac{2r}{\sinh 2r}\,, \\ e^{2h} &= r\coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4}\,, \\ e^{-2\phi} &= e^{-2\phi_0}\frac{2e^h}{\sinh 2r}\,, \end{aligned} \tag{2.4}$$

where ϕ_0 is the value of the dilaton at $r=0$. Near the origin $r=0$ the function e^{2h} behaves as $e^{2h} \sim r^2$ and the metric is non-singular. The solution of the type IIB supergravity includes a Ramond-Ramond three-form $F_{(3)}$ given by

$$F_{(3)} = -\frac{1}{4}(w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4}\sum_a F^a \wedge (w^a - A^a)\,, \tag{2.5}$$

where F^a is the field strength of the su(2) gauge field A^a, defined as:

$$F^a = dA^a + \frac{1}{2}\epsilon_{abc}\, A^b \wedge A^c\,. \tag{2.6}$$

The different components of F^a are:

$$F^1 = -a'\, dr \wedge d\theta\,, \qquad F^2 = a' \sin\theta dr \wedge d\varphi\,, \qquad F^3 = (\,1 - a^2\,)\, \sin\theta d\theta \wedge d\varphi\,, \tag{2.7}$$

where the prime denotes derivative with respect to r. Since $dF_{(3)} = 0$, one can represent $F_{(3)}$ in terms of a two-form potential $C_{(2)}$ as $F_{(3)} = dC_{(2)}$. Actually, it is not difficult to verify that $C_{(2)}$ can be taken as:

$$\begin{aligned} C_{(2)} \;=\; & \frac{1}{4}\Big[\psi\,\Big(\sin\theta d\theta \wedge d\varphi - \sin\tilde\theta d\tilde\theta \wedge d\tilde\varphi\Big) - \cos\theta\cos\tilde\theta d\varphi \wedge d\tilde\varphi\, - \\ & - a\,\Big(d\theta \wedge w^1 - \sin\theta d\varphi \wedge w^2\Big)\Big]\,. \end{aligned} \tag{2.8}$$

Moreover, the equation of motion of $F_{(3)}$ in the Einstein frame is $d(e^{\phi}\,{}^*F_{(3)}) = 0$, where $*$ denotes Hodge duality. Therefore it follows that, at least locally, one must have

$$e^{\phi}\,{}^*F_{(3)} = dC_{(6)}\,, \tag{2.9}$$

with $C_{(6)}$ being a six-form potential. It is readily checked that $C_{(6)}$ can be taken as:

$$C_{(6)} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \mathcal{C}\,, \tag{2.10}$$

where $\mathcal{C}$ is the following two-form:

$$\begin{aligned} \mathcal{C} \;=\; & -\frac{e^{2\phi}}{8}\Big[\Big((a^2-1)a^2\, e^{-2h} - 16\, e^{2h}\Big)\cos\theta d\varphi \wedge dr - (a^2-1)\, e^{-2h}\, w^3 \wedge dr\, + \\ & + a'\,\Big(\sin\theta d\varphi \wedge w^1 + d\theta \wedge w^2\Big)\Big]\,. \end{aligned} \tag{2.11}$$

The Killing spinors ϵ of the above background are worked out in appendix A. They can be obtained by requiring the vanishing of the supersymmetry variations of the fermionic fields of type IIB supergravity. This requirement leads to a system of first-order BPS differential equations (eq. (A.16)) for the functions ϕ, h and a of the ansatz written in eqs. (2.1) and (2.5). It can be easily checked that the functions of eq. (2.4) satisfy the system (A.16). Moreover, it follows from the analysis of appendix A that the Killing spinors ϵ are characterized by the following set of projections:

$$\begin{aligned} \Gamma_{x^0\cdots x^3}(\cos\alpha\,\Gamma_{12} + \sin\alpha\,\Gamma_1\hat\Gamma_2)\,\epsilon \;&=\; \epsilon\,, \\ \Gamma_{12}\,\epsilon \;&=\; \hat\Gamma_{12}\,\epsilon\,, \\ \epsilon \;&=\; i\epsilon^*\,, \end{aligned} \tag{2.12}$$

where the Γ-matrices refer to the frame (A.2) and the explicit expression of the angle α, which depends on the radial coordinate r, is given in eqs. (A.17) and (A.25) (see eqs. (A.3), (A.4) and (A.26)). It is interesting to write here the UV and IR limits of α, namely

$$\lim_{r\to\infty} \alpha = 0\,, \qquad \lim_{r\to 0} \alpha = -\frac{\pi}{2}\,. \tag{2.13}$$

The BPS equations (A.16) also admit a solution in which the function $a(r)$ vanishes, i.e. in which the one-form A^i has only one non-vanishing component, namely A^3. We will refer to this solution as the abelian $\mathcal{N} = 1$ background. Its explicit form can be easily obtained by taking the $r \to \infty$ limit of the functions given in eq. (2.4). Notice that, indeed $a(r) \to 0$ as $r \to \infty$ in eq. (2.4). Neglecting exponentially suppressed terms, one gets:

$$e^{2h} = r - \frac{1}{4}\,, \qquad (a = 0)\,, \tag{2.14}$$

while ϕ can be obtained from the last equation in (2.4). The metric of the abelian background is singular at $r = 1/4$ (the position of the singularity can be moved to $r = 0$ by a redefinition of the radial coordinate). This IR singularity of the abelian background is removed in the non-abelian metric by switching on the A^1, A^2 components of the one-form (2.2). Moreover, when $a = 0$, the angle α appearing in the expression of the Killing spinors (and in the projection (2.12)) is zero, as follows from eq. (A.17).

3. Kappa symmetry

As mentioned in the introduction, the kappa symmetry condition for a supersymmetric embedding of a D5-brane probe is $\Gamma_\kappa\,\epsilon = \epsilon$ (see eq. (1.1)), where ϵ is a Killing spinor of the background. For ϵ such that $\epsilon = i\epsilon^*$ and when there is no worldvolume gauge field, one has:

$$\Gamma_\kappa = \frac{1}{6!}\,\frac{1}{\sqrt{-g}}\,\epsilon^{m_1\cdots m_6}\,\gamma_{m_1\cdots m_6}\,, \tag{3.1}$$

where g is the determinant of the induced metric g_{mn} on the worldvolume

$$g_{mn} = \partial_m X^\mu\,\partial_n X^\nu\,G_{\mu\nu}\,, \tag{3.2}$$

with $G_{\mu\nu}$ being the ten-dimensional metric and $\gamma_{m_1\cdots m_6}$ are antisymmetrized products of worldvolume Dirac matrices γ_m, defined as:

$$\gamma_m = \partial_m X^\mu\,E_\mu^{\underline{\nu}}\,\Gamma_{\underline{\nu}}\,. \tag{3.3}$$

The vierbeins $E_\mu^{\underline{\nu}}$ are the coefficients which relate the one-forms $e^{\underline{\nu}}$ of the frame and the differentials of the coordinates, i.e. $e^{\underline{\nu}} = E_\mu^{\underline{\nu}} dX^\mu$. Let us take as worldvolume coordinates $(x^0, \cdots, x^3, \theta, \varphi)$. Then, for an embedding with $\tilde\theta = \tilde\theta(\theta,\varphi)$, $\tilde\varphi = \tilde\varphi(\theta,\varphi)$, $\psi = \psi(\theta,\varphi)$ and $r = r(\theta,\varphi)$, the kappa symmetry matrix Γ_κ takes the form:

$$\Gamma_\kappa = \frac{e^\phi}{\sqrt{-g}}\,\Gamma_{x^0\cdots x^3}\,\gamma_{\theta\varphi}\,, \tag{3.4}$$

JHEP12(2003)024

with $\gamma_{\theta\varphi}$ being the antisymmetrized product of the two induced matrices γ_θ and γ_φ, which can be written as:

$$\begin{aligned} e^{-\frac{\phi}{4}}\gamma_\theta &= e^h\,\Gamma_1 + (V_{1\theta}+\frac{a}{2})\,\hat\Gamma_1 + V_{2\theta}\,\hat\Gamma_2 + V_{3\theta}\,\hat\Gamma_3 + \partial_\theta r\Gamma_r\,,\\ \frac{e^{-\frac{\phi}{4}}}{\sin\theta}\gamma_\varphi &= e^h\,\Gamma_2 + V_{1\varphi}\,\hat\Gamma_1 + (V_{2\varphi}-\frac{a}{2})\,\hat\Gamma_2 + V_{3\varphi}\,\hat\Gamma_3 + \frac{\partial_\varphi r}{\sin\theta}\,\Gamma_r\,, \end{aligned} \tag{3.5}$$

where the V's can be obtained by computing the pullback on the worldvolume of the left invariant one-forms w^i, and are given by:

$$\begin{aligned} V_{1\theta} &= \frac{1}{2}\,\cos\psi\,\partial_\theta\,\tilde\theta + \frac{1}{2}\sin\psi\sin\tilde\theta\,\partial_\theta\,\tilde\varphi\,,\\ \sin\theta\,V_{1\varphi} &= \frac{1}{2}\,\cos\psi\,\partial_\varphi\,\tilde\theta + \frac{1}{2}\sin\psi\sin\tilde\theta\,\partial_\varphi\,\tilde\varphi\,,\\ V_{2\theta} &= -\frac{1}{2}\,\sin\psi\,\partial_\theta\,\tilde\theta + \frac{1}{2}\cos\psi\sin\tilde\theta\,\partial_\theta\,\tilde\varphi\,,\\ \sin\theta\,V_{2\varphi} &= -\frac{1}{2}\,\sin\psi\,\partial_\varphi\,\tilde\theta + \frac{1}{2}\cos\psi\sin\tilde\theta\,\partial_\varphi\,\tilde\varphi\,,\\ V_{3\theta} &= \frac{1}{2}\,\partial_\theta\psi + \frac{1}{2}\cos\tilde\theta\,\partial_\theta\,\tilde\varphi\,,\\ \sin\theta\,V_{3\varphi} &= \frac{1}{2}\,\partial_\varphi\psi + \frac{1}{2}\cos\tilde\theta\,\partial_\varphi\,\tilde\varphi + \frac{1}{2}\,\cos\theta\,. \end{aligned} \tag{3.6}$$

By using the projections (A.3) and (A.8) one can compute the action of $\gamma_{\theta\varphi}$ on the Killing spinor ϵ. It is clear that one arrives at an expression of the type:

$$\frac{e^{-\frac{\phi}{2}}}{\sin\theta}\gamma_{\theta\varphi}\epsilon = \Big[c_{12}\Gamma_{12} + c_{1\hat2}\Gamma_1\hat\Gamma_2 + c_{1\hat1}\Gamma_1\hat\Gamma_1 + c_{1\hat3}\Gamma_1\hat\Gamma_3 + c_{\hat1\hat3}\hat\Gamma_{13} + c_{\hat2\hat3}\hat\Gamma_{23} + c_{2\hat3}\Gamma_2\hat\Gamma_3\Big]\,\epsilon\,, \tag{3.7}$$

where the c's are coefficients that can be explicitly computed. By using eq. (3.7) we can obtain the action of Γ_κ on ϵ and we can use this result to write the kappa symmetry projection $\Gamma_\kappa\epsilon = \epsilon$. Actually, eq. (1.1) is automatically satisfied if it reduces to the first equation in eq. (2.12). If we want this to happen, all terms except the ones containing $\Gamma_{12}\,\epsilon$ and $\Gamma_1\,\hat\Gamma_2\,\epsilon$ on the right-hand side of eq. (3.7) should vanish. Then, we should require

$$c_{1\hat1} = c_{1\hat3} = c_{\hat1\hat3} = c_{\hat2\hat3} = c_{2\hat3} = 0\,. \tag{3.8}$$

By using the explicit expressions of the c's one can obtain from eq. (3.8) five conditions that our supersymmetric embeddings must necessarily satisfy. These conditions are:

$$e^h\,(V_{1\varphi}+V_{2\theta}) = 0\,, \tag{3.9}$$

$$e^h(V_{3\varphi}+\cos\alpha\partial_\theta r) + \left(V_{2\varphi}-\frac{a}{2}\right)\sin\alpha\partial_\theta r - V_{2\theta}\sin\alpha\frac{\partial_\varphi r}{\sin\theta} = 0\,, \tag{3.10}$$

$$\left(V_{1\theta}+\frac{a}{2}\right)V_{3\varphi} - V_{3\theta}V_{1\varphi} - e^h\sin\alpha\partial_\theta r + \left(V_{2\varphi}-\frac{a}{2}\right)\cos\alpha\partial_\theta r - V_{2\theta}\cos\alpha\frac{\partial_\varphi r}{\sin\theta} = 0\,, \tag{3.11}$$

$$V_{3\varphi}V_{2\theta} - V_{3\theta}\left(V_{2\varphi}-\frac{a}{2}\right) - V_{1\varphi}\cos\alpha\partial_\theta r + \left(e^h\sin\alpha + \left(V_{1\theta}+\frac{a}{2}\right)\cos\alpha\right)\frac{\partial_\varphi r}{\sin\theta} = 0\,, \tag{3.12}$$

$$\sin\alpha V_{1\varphi}\partial_\theta r - e^hV_{3\theta} + \left(e^h\cos\alpha - \left(V_{1\theta}+\frac{a}{2}\right)\sin\alpha\right)\frac{\partial_\varphi r}{\sin\theta} = 0\,. \tag{3.13}$$

Moreover, if we want the kappa symmetry projection $\Gamma_\kappa \epsilon = \epsilon$ to coincide with the SUGRA projection, the ratio of the coefficients of the terms with $\Gamma_1 \hat{\Gamma}_2 \epsilon$ and $\Gamma_{12}\epsilon$ must be $\tan\alpha$, i.e. one must have:

$$\tan\alpha = \frac{c_{1\hat{2}}}{c_{12}}\,. \tag{3.14}$$

The explicit form of c_{12} and $c_{1\hat{2}}$ is:

$$\begin{aligned} c_{12} &= e^{2h} + V_{1\theta}\,V_{2\varphi} - V_{2\theta}\,V_{1\varphi} - \frac{a}{2}\,(V_{1\theta} - V_{2\varphi}) - \frac{a^2}{4} - \cos\alpha V_{3\varphi}\,\partial_\theta r + \cos\alpha V_{3\theta}\frac{\partial_\varphi r}{\sin\theta}\,,\\ c_{1\hat{2}} &= e^h\,(V_{2\varphi} - V_{1\theta} - a) - \sin\alpha V_{3\varphi}\,\partial_\theta r + \sin\alpha V_{3\theta}\,\frac{\partial_\varphi r}{\sin\theta}\,. \end{aligned} \tag{3.15}$$

Amazingly, except when r is constant and takes values in the interval $0 < r < \infty$ (see section 4), eq. (3.14) is a consequence of eqs. (3.9)–(3.13). Actually, by eliminating $V_{3\theta}$ of eqs. (3.12) and (3.13), and making use of eqs. (3.9) and (3.10), one arrives at the following expression of $\tan\alpha$:

$$\tan\alpha = \frac{e^h\,(V_{2\varphi} - V_{1\theta} - a)}{e^{2h} + V_{1\theta}\,V_{2\varphi} - V_{2\theta}\,V_{1\varphi} - \frac{a}{2}\,(V_{1\theta} - V_{2\varphi}) - \frac{a^2}{4}}\,. \tag{3.16}$$

Notice that the terms of $c_{1\hat{2}}$ (c_{12}) which do not contain $\sin\alpha$ ($\cos\alpha$) are just the ones in the numerator (denominator) of the right-hand side of this equation. It follows from this fact that eq. (3.14) is satisfied if eqs. (3.9)–(3.13) hold. Moreover, by using the values of $\cos\alpha$ and $\sin\alpha$ given in eq. (A.17), one obtains the interesting relation:

$$(1 + a^2 + 4e^{2h})\,(V_{1\theta} - V_{2\varphi}) = 4a\left(V_{2\theta}^2 + V_{1\theta}\,V_{2\varphi} - \frac{1}{4}\right). \tag{3.17}$$

The system of eqs. (3.9)–(3.13) is rather involved and, although it could seem at first sight very difficult and even hopeless to solve, we will be able to do it in some particular cases. Moreover, it is interesting to notice that, by simple manipulations, one can obtain the following expressions of the partial derivatives of r:

$$\begin{aligned} \partial_\theta r &= -\cos\alpha V_{3\varphi} + \sin\alpha\, e^{-h}\left[\left(V_{1\theta} + \frac{a}{2}\right) V_{3\varphi} - V_{3\theta}\,V_{1\varphi}\right],\\ \partial_\varphi r &= \cos\alpha\sin\theta V_{3\theta} + \sin\alpha\,\sin\theta e^{-h}\left[\left(V_{2\varphi} - \frac{a}{2}\right) V_{3\theta} - V_{3\varphi}\,V_{2\theta}\right], \end{aligned} \tag{3.18}$$

which will be very useful in our analysis.

4. Branes wrapped at fixed distance

In this section we will consider the possibility of wrapping the D5-branes at a fixed distance $r > 0$ from the origin. It is clear that, in this case, we have $\partial_\theta r = \partial_\varphi r = 0$ and many of the terms on the left-hand side of eqs. (3.9)–(3.13) cancel. Moreover e^h is non-vanishing when $r > 0$ and it can be factored out in these equations. Thus, the equations (3.9)–(3.13) of kappa symmetry when the radial coordinate r is constant and non-zero reduce to:

$$V_{1\varphi} + V_{2\theta} = V_{3\varphi} = V_{3\theta} = 0\,. \tag{4.1}$$

From the equations $V_{3\varphi} = V_{3\theta} = 0$ we obtain the following differential equations for ψ

$$\partial_\theta\, \psi = -\cos\tilde\theta\, \partial_\theta \tilde\varphi\,, \qquad \partial_\varphi\, \psi = -\cos\tilde\theta\, \partial_\varphi\tilde\varphi - \cos\theta\,. \tag{4.2}$$

The integrability condition for this system gives:

$$\partial_\varphi \tilde\theta\, \partial_\theta \tilde\varphi - \partial_\theta\tilde\theta\, \partial_\varphi\tilde\varphi = \frac{\sin\theta}{\sin\tilde\theta}\,. \tag{4.3}$$

By using this condition and the definition of the V's (eq. (3.6)) one can prove that

$$V_{1\theta}\, V_{2\varphi} - V_{1\varphi}\, V_{2\theta} = -\frac{1}{4}\,. \tag{4.4}$$

Let us now define Δ as follows:

$$V_{2\varphi} - V_{1\theta} \equiv \Delta\,. \tag{4.5}$$

By using the expression of the V's in terms of the angles, one can combine eq. (4.5) and the condition $V_{1\varphi} + V_{2\theta} = 0$ in the following matrix equation

$$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \sin\theta\partial_\theta\tilde\theta - \sin\tilde\theta\partial_\varphi\tilde\varphi \\ \sin\theta\sin\tilde\theta\partial_\theta\tilde\varphi + \partial_\varphi\tilde\theta \end{pmatrix} = \begin{pmatrix} -2\Delta\sin\theta \\ 0 \end{pmatrix}\,. \tag{4.6}$$

Since the matrix appearing on the left-hand side is non-singular, we can multiply by its inverse. By doing this one arrives at the following equations:

$$\begin{aligned} \partial_\theta\tilde\theta - \frac{\sin\tilde\theta}{\sin\theta}\, \partial_\varphi\tilde\varphi \;&=\; -2\Delta\, \cos\psi\,, \\ \frac{\partial_\varphi\tilde\theta}{\sin\theta} + \sin\tilde\theta\partial_\theta\tilde\varphi \;&=\; -2\Delta\sin\psi\,. \end{aligned} \tag{4.7}$$

Substituting the derivatives of $\tilde\theta$ obtained from the above equations into the integrability condition (4.3) we obtain after some calculation

$$\sin^2\tilde\theta\, \left(\partial_\theta\tilde\varphi + \Delta\,\frac{\sin\psi}{\sin\tilde\theta}\right)^2 + \frac{\sin^2\tilde\theta}{\sin^2\theta}\, \left(\partial_\varphi\tilde\varphi - \Delta\,\cos\psi\frac{\sin\theta}{\sin\tilde\theta}\right)^2 = \Delta^2 - 1\,. \tag{4.8}$$

The right-hand side of eq. (4.8) is non-negative. Then, one obtains a bound for Δ:

$$\Delta^2 \geq 1\,. \tag{4.9}$$

Notice that we have not imposed all the requirements of kappa symmetry. Indeed, it still remains to check that the ratios between the coefficients c_{12} and $c_{1\hat 2}$ is the one corresponding to the projection of the background. Using eq. (4.4) and the definition of Δ (eq. (4.5)), one obtains:

$$c_{12} = e^{2h} + a\,\frac{\Delta}{2} - \frac{a^2+1}{4}\,, \qquad c_{1\hat 2} = e^h\,(\Delta - a)\,. \tag{4.10}$$

Then, one must have:

$$\tan\alpha = \frac{e^h\,(\Delta - a)}{e^{2h} + a\,\frac{\Delta}{2} - \frac{a^2+1}{4}} = -\frac{ae^h}{e^{2h} + \frac{1-a^2}{4}}\,, \tag{4.11}$$

where we have used the values of $\sin\alpha$ and $\cos\alpha$ given in the appendix A (eq. (A.17)). If e^h is nonzero (and finite), we can factor it out in eq. (4.11) and obtain the following expression of Δ:

$$\Delta = \frac{2a}{1+a^2+4e^{2h}}\,. \tag{4.12}$$

Notice that Δ depends only on the coordinate r and is a monotonically decreasing function such that $0<\Delta<1$ for $0<r<\infty$ and

$$\lim_{r\to 0}\Delta = 1\,, \qquad \lim_{r\to\infty}\Delta = 0\,. \tag{4.13}$$

As $\Delta<1$, the bound (4.9) is not satisfied and, thus, there is no solution to our equations for $0<r<\infty$. Notice that this was to be expected from the lack of moduli space of the $\mathcal{N}=1$ theories.

Let us now consider the possibility of placing the brane probe at $r\to\infty$. Notice that in this case eq. (4.11) is satisfied for any finite value of Δ. However, the value $\Delta=1$ is special since, in this case, the right-hand side of eq. (4.8) vanishes and we obtain two equations that determine the derivatives of φ, namely:

$$\partial_\theta\tilde\varphi = -\frac{\sin\psi}{\sin\tilde\theta}\,, \qquad \partial_\varphi\tilde\varphi = \cos\psi\,\frac{\sin\theta}{\sin\tilde\theta}\,. \tag{4.14}$$

Using these equations into the system (4.7) for $\Delta=1$ one gets the following equations for the derivatives of $\tilde\theta$:

$$\partial_\theta\tilde\theta = -\cos\psi\,, \qquad \partial_\varphi\tilde\theta = -\sin\theta\sin\psi\,, \tag{4.15}$$

and, similarly, the equations (4.2) for ψ become:

$$\partial_\theta\psi = \sin\psi\cot\tilde\theta\,, \qquad \partial_\varphi\psi = -\sin\theta\cot\tilde\theta\,\cos\psi - \cos\theta\,. \tag{4.16}$$

The equations (4.14) and (4.15) can be regarded as coming from the following identifications of the frame forms in the (θ,φ) and $(\tilde\theta,\tilde\varphi)$ spheres:

$$\begin{pmatrix} d\tilde\theta \\ \sin\tilde\theta d\tilde\varphi \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} -d\theta \\ \sin\theta d\varphi \end{pmatrix}. \tag{4.17}$$

The differential equations (4.16) are just the integrability conditions of the system (4.17). Another interesting observation is that one can prove by using the differential eqs. (4.14)–(4.16) that the pullbacks of the su(2) left-invariant one-forms are

$$P[w^1] = -d\theta\,, \qquad P[w^2] = \sin\theta d\varphi\,, \qquad P[w^3] = -\cos\theta d\varphi\,. \tag{4.18}$$

Let us try to find a solution of the differential equations (4.14)–(4.16) in which $\tilde\theta=\tilde\theta(\theta)$ and $\tilde\varphi=\tilde\varphi(\varphi)$. The vanishing of $\partial_\varphi\tilde\theta$ and $\partial_\theta\tilde\varphi$ immediately leads to $\sin\psi=0$ or $\psi=0,\pi$ (mod 2π). Thus ψ is constant in this case. Let us put $\cos\psi=\eta=\pm1$. The vanishing of $\partial_\theta\psi$ is automatic, whereas the condition $\partial_\varphi\psi=0$ leads to a relation between $\tilde\theta$ and θ:

$$\cot\tilde\theta = -\eta\cot\theta\,. \tag{4.19}$$

In the case $\psi = 0$, one has $\eta = 1$ and the previous relation yields $\tilde{\theta} = \pi - \theta$. Notice that this relation is in agreement with the first equation in eq. (4.15). Moreover, the second equation in (4.14) gives $\tilde{\varphi} = \varphi$. Similarly one can solve the equations for $\psi = \pi$. The solutions in these two cases are just the ones used in ref. [10] in the calculation of the beta function (with some correction in the $\psi = 0$ case to have the correct range of θ and $\tilde{\theta}$), namely:

$$\begin{aligned} &\tilde{\theta} = \pi - \theta\,, && \tilde{\varphi} = \varphi\,, && \psi = 0 \pmod{2\pi}\,, \\ &\tilde{\theta} = \theta\,, && \tilde{\varphi} = 2\pi - \varphi\,, && \psi = \pi \pmod{2\pi}\,. \end{aligned} \tag{4.20}$$

It follows from our results that the embedding of ref. [10] is only supersymmetric asymptotically when $r \to \infty$. In this sense, although it is somehow distinguished, it is not unique since for any embedding such that the V's are finite when $r \to \infty$, the determinant of the induced metric diverges as $\sqrt{-g} \sim e^{\frac{3\phi}{2}+2h}$ and the only term which survives in the equation $\Gamma_\kappa \epsilon = \epsilon$ is the one with the matrix Γ_{12}, giving rise to the same projection as the background for $r \to \infty$.

5. Worldvolume solitons (abelian case)

Let us consider the case $a = \alpha = 0$ in the general equations of section 3. From equations (3.9) and (3.17) we get the following (Cauchy-Riemann like) equations:

$$V_{1\theta} = V_{2\varphi}\,, \qquad V_{1\varphi} = -V_{2\theta}\,, \tag{5.1}$$

whereas, from eq. (3.18) we obtain that the derivatives of r are given by:

$$r_\theta = -V_{3\varphi}\,, \qquad r_\varphi = \sin\theta\, V_{3\theta}\,, \tag{5.2}$$

where $r_\theta \equiv \partial_\theta r$ and $r_\varphi \equiv \partial_\varphi r$. It can be easily demonstrated that, in this abelian case, the full set of equations (3.9)–(3.13) collapses to the two pairs of equations (5.1) and (5.2). Notice that $c_{1\hat{2}} = 0$ when $a = \alpha = 0$ and eq. (5.1) holds and, thus, eq. (3.14) is satisfied identically.

Let us study first the two equations (5.1). By using the same technique as the one employed in section 4 to derive eq. (4.7), it can be shown easily that they can be written as

$$\sin\theta \partial_\theta \tilde{\theta} = \sin\tilde{\theta} \partial_\varphi \tilde{\varphi}\,, \qquad \partial_\varphi \tilde{\theta} = -\sin\theta \sin\tilde{\theta} \partial_\theta \tilde{\varphi}\,. \tag{5.3}$$

In order to find the general solution of eq. (5.3), let us introduce a new set of variables u and $\tilde{u}$ as follows:

$$u = \log\tan\frac{\theta}{2}\,, \qquad \tilde{u} = \log\tan\frac{\tilde{\theta}}{2}\,. \tag{5.4}$$

Then, eq. (5.3) can be written as the Cauchy-Riemann equations in the (u, φ) and $(\tilde{u}, \tilde{\varphi})$ variables, namely:

$$\frac{\partial \tilde{u}}{\partial u} = \frac{\partial \tilde{\varphi}}{\partial \varphi}\,, \qquad \frac{\partial \tilde{u}}{\partial \varphi} = -\frac{\partial \tilde{\varphi}}{\partial u}\,. \tag{5.5}$$

Since $u, \tilde{u} \in (-\infty, +\infty)$ and $\varphi, \tilde{\varphi} \in (0, 2\pi)$, the above equations are the Cauchy-Riemann equations in a band. The general solution of these equations is of the form:

$$\tilde{u} + i\tilde{\varphi} = f(u + i\varphi)\,, \tag{5.6}$$

where f is an arbitrary function. Given any function f, it is clear that the above equation provides the general solution $\tilde{\theta}(\theta,\varphi)$ and $\tilde{\varphi}(\theta,\varphi)$ of the system (5.3).

Let us turn now to the analysis of the system of equations (5.2), which determines the radial coordinate r. By using the explicit values of $V_{3\varphi}$ and $V_{3\theta}$, these equations can be written as:

$$\begin{aligned} r_\theta &= -\frac{1}{2\sin\theta}\,\partial_\varphi\,\psi - \frac{1}{2}\frac{\cos\tilde{\theta}}{\sin\theta}\,\partial_\varphi\tilde{\varphi} - \frac{1}{2}\cot\theta\,, \\ r_\varphi &= \frac{\sin\theta}{2}\,\partial_\theta\,\psi + \frac{\sin\theta}{2}\,\cos\tilde{\theta}\,\partial_\theta\tilde{\varphi}\,, \end{aligned} \tag{5.7}$$

where $\tilde{\theta}(\theta,\varphi)$ and $\tilde{\varphi}(\theta,\varphi)$ are solutions of eq. (5.3). In terms of the derivatives with respect to variable u defined above ($\sin\theta\partial_\theta = \partial_u$), these equations become:

$$\begin{aligned} r_u &= -\frac{1}{2}\,\partial_\varphi\psi - \frac{1}{2}\,\cos\tilde{\theta}\,\partial_\varphi\tilde{\varphi} - \frac{1}{2}\,\cos\theta\,, \\ r_\varphi &= \frac{1}{2}\,\partial_u\psi + \frac{1}{2}\,\cos\tilde{\theta}\,\partial_u\tilde{\varphi}\,. \end{aligned} \tag{5.8}$$

The integrability condition of these equations is just $\partial_\varphi r_u = \partial_u r_\varphi$. As any solution $(\tilde{\theta}, \tilde{\varphi})$ of the Cauchy-Riemann equations (5.3) satisfies:

$$\partial_\varphi\tilde{\theta}\partial_\varphi\tilde{\varphi} = -\partial_u\tilde{\theta}\partial_u\tilde{\varphi}\,, \tag{5.9}$$

and, since $\tilde{\varphi}$, being a solution of the Cauchy-Riemann equations, is harmonic in (u,φ), it follows that $\partial_\varphi r_u = \partial_u r_\varphi$ if and only if ψ is also harmonic in (u,φ), i.e. the differential equation for ψ is just the Laplace equation in the (u,φ) plane, namely

$$\partial_\varphi^2\psi + \partial_u^2\psi = 0\,. \tag{5.10}$$

Remarkably, the form of $r(\theta,\varphi)$ can be obtained in general. Let us define:

$$\Lambda(\theta,\varphi) = \int_0^\varphi d\varphi\,\sin\theta\partial_\theta\psi(\theta,\varphi) - \int \frac{d\theta}{\sin\theta}\,\partial_\varphi\psi(\theta,0)\,. \tag{5.11}$$

It follows from this definition and the fact that ψ is harmonic in (u,φ) that ψ and Λ also satisfy the Cauchy-Riemann equations:

$$\frac{\partial\Lambda}{\partial\varphi} = \frac{\partial\psi}{\partial u}\,, \qquad \frac{\partial\Lambda}{\partial u} = -\frac{\partial\psi}{\partial\varphi}\,. \tag{5.12}$$

Thus ψ and Λ are conjugate harmonic functions, i.e. $\psi + i\Lambda$ is an analytic function of $u + i\varphi$. Notice that given Λ one can obtain ψ by integrating the previous differential equations. It can be checked by using the Cauchy-Riemann equations that the derivatives of r, as given by the right hand side of eq. (5.8), can be written as $r_\theta = \partial_\theta F$, $r_\varphi = \partial_\varphi F$, where:

$$F(\theta,\varphi) = \frac{1}{2}\left[\Lambda(\theta,\varphi) - \log(\sin\theta\sin\tilde{\theta}(\theta,\varphi))\right]. \tag{5.13}$$

Therefore, it follows that:

$$e^{2r} = C\frac{e^{\Lambda(\theta,\varphi)}}{\sin\theta\sin\tilde{\theta}(\theta,\varphi)}, \tag{5.14}$$

with C being a constant. We will make use of this amazingly simple expression to derive the equation of some particularly interesting embeddings.

5.1 n-Winding solitons

First of all, let us consider the particular class of solutions of the Cauchy-Riemann eqs. (5.5):

$$\tilde{u} + i\tilde{\varphi} = n(u + i\varphi) + \text{constant}, \tag{5.15}$$

where n is an integer and the constant is complex. In terms of the original variables:

$$\tan\frac{\tilde{\theta}}{2} = \tilde{c}\left(\tan\frac{\theta}{2}\right)^n, \qquad \tilde{\varphi} = n\,\varphi + \varphi_0, \tag{5.16}$$

with $\tilde{c}$ and φ_0 being constants. It is clear that in this solution the $\tilde{\varphi}$ coordinate of the probe wraps n times the $[0, 2\pi]$ interval as φ varies between 0 and 2π. Let us now assume that the coordinate ψ is constant, i.e. $\psi = \psi_0$. It is clear from its definition that the function $\Lambda(\theta,\varphi)$ is zero in this case. Moreover, by using the identities

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \qquad \tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}}, \tag{5.17}$$

one can prove that:

$$\sin\tilde{\theta} = 2\sqrt{c}\,\frac{(\sin\theta)^n}{(1+\cos\theta)^n + c\,(1-\cos\theta)^n}, \tag{5.18}$$

where $c = \tilde{c}^2$. After plugging this result in eq. (5.14), one obtains the explicit form of the function $r(\theta)$, namely:

$$e^{2r} = \frac{e^{2r_*}}{1+c}\,\frac{(1+\cos\theta)^n + c\,(1-\cos\theta)^n}{(\sin\theta)^{n+1}}, \tag{5.19}$$

where $r_* = r(\theta = \pi/2)$. We will call n-winding embedding to the brane configuration corresponding to eqs. (5.16) and (5.19) for a constant value of the angle ψ.

Let us pause for a moment to study the function (5.19). First of all it is easy to verify that this function is invariant if we change $n \to -n$ and $c \to 1/c$ (or equivalently changing $\theta \to \pi - \theta$ for the same constant c). Actually, in what follows we shall take the integration constant $c = 1$ and thus we can restrict ourselves to the case in which n is non-negative. In this $c = 1$ case r_* is the minimal separation between the brane probe and the origin. Another observation is that r diverges for $\theta = 0, \pi$, which corresponds to the location of

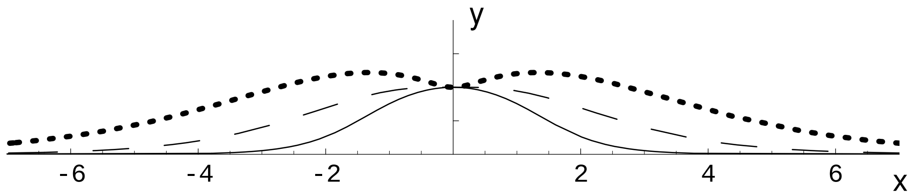

Figure 1: Curves $y = y(x)$ for three values of the winding number n: $n = 0$ (solid line), $n = 1$ (dashed curve) and $n = 2$ (dotted line). These three curves correspond to $r_* = 1$.

the spikes of the worldvolume solitons. Therefore the supersymmetric embedding we have found is non-compact. Actually, it has the topology of a cylinder whose compact direction is parametrized by φ. This cylinder connects the two poles at $\theta = 0, \pi$ of the (θ, φ) sphere at $r = \infty$ and passes at a distance r_* from the origin.

It is also interesting to discuss the symmetries of our solutions. Recall that the angle ψ is constant for our embeddings. Thus, it is clear that one can shift it by an arbitrary constant ϵ as $\psi \to \psi + \epsilon$. This U(1) symmetry corresponds to an isometry of the abelian background which quantum-mechanically is broken to $\mathbb{Z}_{2N}$ as a consequence of the flux quantization of the RR two-form potential [6, 34, 35]. In the gauge theory side this isometry has been identified [6, 34, 35] with the U(1) R-symmetry of the $\mathcal{N} = 1$ SYM theory, which is broken down to $\mathbb{Z}_{2N}$ by a field theory anomaly [29]. On the other hand, it is also clear that we have an additional U(1) associated to constant shifts in $\tilde{\varphi}$, which are equivalent to a redefinition of φ_0 in eq. (5.16).

To visualize the shape of the brane in these solutions it is rather convenient to introduce the following cartesian coordinates x and y

$$x = r \cos\theta \,, \qquad y = r \sin\theta \,. \tag{5.20}$$

In terms of (x, y) the D5-brane embedding will be described by means of a curve $y = y(x)$. Notice that $y \geq 0$, whereas $-\infty < x < +\infty$. The value of the coordinate y at $x = 0$ is just r_*, i.e. $y(x = 0) = r_*$. Moreover, for large values of the coordinate x, the function $y(x) \to 0$ exponentially as

$$y(x) \approx C\,|x|\, e^{-\frac{2}{|n|+1}|x|} \,, \qquad (|x| \to \infty) \,, \tag{5.21}$$

where C is a constant. To illustrate this behaviour we have plotted in figure 1 the curves $y(x)$ for three different values of the winding n and the same value of r_*.

A particularly interesting case is obtained when $n = \pm 1$. By adjusting properly the constant φ_0 in eq. (5.16) the angular embedding reduces to:

$$\begin{aligned}
&\tilde{\theta} = \theta \,, && \tilde{\varphi} = \varphi + \text{constant} \,, && (n = 1) \,, \\
&\tilde{\theta} = \pi - \theta \,, && \tilde{\varphi} = 2\pi - \varphi + \text{constant} \,, && (n = -1) \,,
\end{aligned} \tag{5.22}$$

with ψ being constant. These types of angular embeddings are similar to the ones considered in ref. [10] (although they are not the same, see eq. (4.20)) and we will refer to them as unit-winding embeddings. Notice that the two cases displayed in eq. (5.22) represent the two possible identifications of the (θ, φ) and $(\tilde{\theta}, \tilde{\varphi})$ two-spheres.

When $n = 0$ the brane is wrapping the (θ, φ) sphere at constant values of $\tilde{\theta}$ and $\tilde{\varphi}$, i.e. one has:

$$\tilde{\theta} = \text{constant} = \tilde{\theta}_0 \,, \quad \tilde{\varphi} = \text{constant} = \tilde{\varphi}_0 \,, \qquad (n = 0) \,. \tag{5.23}$$

We will refer to this case as zero-winding embedding [9].

One can verify that the brane embeddings we have found are solutions of the probe equations of motion. Actually, they are supersymmetric worldvolume solitons of the D5 brane probe. To illustrate this fact let us show that these configurations saturate a BPS energy bound. To simplify matters, let us assume that the angular embedding is the one displayed in eq. (5.16) and let $r(\theta)$ be an arbitrary function. The Dirac-Born-Infeld (DBI) lagrangian density for a D5-brane with unit tension is:

$$\mathcal{L} = -e^{-\phi} \sqrt{-g_{st}} + P[C_{(6)}] \,, \tag{5.24}$$

where g_{st} is the determinant of the induced metric in the string frame ($g_{st} = e^{3\phi}\, g$) and $P[C_{(6)}]$ is the pullback on the worldvolume of the RR six-form of the background. The elements of the induced metric for the n-winding solution along the angular coordinates are:

$$\begin{aligned} g_{\theta\theta} &= e^{\frac{\phi}{2}} \left(e^{2h} + \frac{n^2}{4} \frac{\sin^2\tilde{\theta}}{\sin^2\theta} + r_\theta^2 \right) \,, \\ g_{\varphi\varphi} &= e^{\frac{\phi}{2}} \left(e^{2h} + \frac{n^2}{4} \frac{\sin^2\tilde{\theta}}{\sin^2\theta} + V_{3\varphi}^2 \right) \sin^2\theta \,. \end{aligned} \tag{5.25}$$

From this expression one immediately obtains the determinant of the induced metric, namely:

$$\sqrt{-g} = e^{\frac{3\phi}{2}} \sin\theta \sqrt{\left(e^{2h} + \frac{n^2}{4} \frac{\sin^2\tilde{\theta}}{\sin^2\theta} + V_{3\varphi}^2 \right) \left(e^{2h} + \frac{n^2}{4} \frac{\sin^2\tilde{\theta}}{\sin^2\theta} + r_\theta^2 \right)} \,. \tag{5.26}$$

Moreover, the pullback on the worldvolume of the two-form $\mathcal{C}$ is:[1]

$$P[\mathcal{C}] = \frac{e^{2\phi}}{8} \left(16 e^{2h} \cos\theta - n e^{-2h} \cos\tilde{\theta} \right) r_\theta \, d\varphi \wedge d\theta \,. \tag{5.27}$$

The hamiltonian density $\mathcal{H}$ for a static configuration is just $\mathcal{H} = -\mathcal{L}$ or:

$$\begin{aligned} \mathcal{H} = e^{2\phi} \Bigg[& \sin\theta \sqrt{\left(e^{2h} + \frac{n^2}{4} \frac{\sin^2\tilde{\theta}}{\sin^2\theta} + V_{3\varphi}^2 \right) \left(e^{2h} + \frac{n^2}{4} \frac{\sin^2\tilde{\theta}}{\sin^2\theta} + r_\theta^2 \right)} - \\ & - \frac{1}{8} \left(16 e^{2h} \cos\theta - n e^{-2h} \cos\tilde{\theta} \right) r_\theta \Bigg] \,. \end{aligned} \tag{5.28}$$

[1] It is worth mentioning that the pullback of the RR two-form to the worldvolume is

$$P[C_{(2)}] = \frac{\psi}{4} \, d\varphi \wedge (n \sin\tilde{\theta} d\tilde{\theta} - \sin\theta d\theta) \,,$$

where $\tilde{\theta}(\theta)$ is the function displayed in eq. (5.18) . From this expression it is straightforward to verify that the RR two-form flux through the two-submanifold where we are wrapping our brane is

$$\int P[C_{(2)}] = \pi\psi\,(|n| - 1) \,,$$

and thus it vanishes iff $n = \pm 1$.

It can be checked that, for an arbitrary function $r(\theta)$, one can write $\mathcal{H}$ as:

$$\mathcal{H} = \mathcal{Z} + \mathcal{S}\,, \tag{5.29}$$

where $\mathcal{Z}$ is a total derivative:

$$\mathcal{Z} = -\partial_\theta \left[e^{2\phi} \left(e^{2h}\,\cos\theta + \frac{n}{4}\,\cos\tilde{\theta} \right) \right], \tag{5.30}$$

and $\mathcal{S}$ is non-negative:

$$\mathcal{S} \geq 0\,, \tag{5.31}$$

with $\mathcal{S} = 0$ precisely when the BPS equations for the embedding are satisfied. The expression of $\mathcal{S}$ is:

$$\begin{aligned} \mathcal{S} \,=\, \sin\theta\, e^{2\phi} \Bigg[& \sqrt{\left(e^{2h} + \frac{n^2}{4}\frac{\sin^2\tilde{\theta}}{\sin^2\theta} + V_{3\varphi}^2 \right)\left(e^{2h} + \frac{n^2}{4}\,\frac{\sin^2\tilde{\theta}}{\sin^2\theta} + r_\theta^2 \right)} - \\ & - \left(e^{2h} + \frac{n^2}{4}\,\frac{\sin^2\tilde{\theta}}{\sin^2\theta} - V_{3\varphi}\, r_\theta \right) \Bigg]\,. \end{aligned} \tag{5.32}$$

The BPS equation for r in this case is $r_\theta = -V_{3\varphi}$ (see eq. (5.2)). If this equation is satisfied, the first term on the right-hand side of eq. (5.32) is a square root of a perfect square which cancels against the second term of this equation. Moreover, it is easy to check that the condition $\mathcal{S} \geq 0$ is equivalent to:

$$(r_\theta + V_{3\varphi})^2 \geq 0\,, \tag{5.33}$$

which is obviously satisfied and reduces to an equality if and only if the BPS equation for the embedding is satisfied.

5.2 (n,m)-Winding solitons

The solutions found in the previous section are easily generalized if we allow the angle ψ to wind a certain number of times as the coordinate φ varies from $\varphi = 0$ to $\varphi = 2\pi$. Recalling that ψ ranges from 0 to 4π, let us write the following ansatz for $\psi(\varphi)$:

$$\psi = \psi_0 + 2m\varphi\,, \tag{5.34}$$

where m is an integer. It is obvious that the above function satisfies the Laplace equation (5.10). Moreover, its harmonic conjugate Λ is immediately obtained by solving the Cauchy-Riemann differential equations (5.12), namely:

$$\Lambda = -2mu\,. \tag{5.35}$$

In terms of the angle θ, the above equation becomes:

$$e^\Lambda = \frac{1}{(\tan\frac{\theta}{2})^{2m}}\,. \tag{5.36}$$

By plugging this result in eq. (5.14), and using the value of $\sin\tilde{\theta}$ given in eq. (5.18), it is straightforward to obtain the function $r(\theta)$ of the embedding. One gets:

$$e^{2r} = \frac{e^{2r_*}}{1+c}\,\frac{(1+\cos\theta)^n + c\,(1-\cos\theta)^n}{[\tan\frac{\theta}{2}]^{2m}(\sin\theta)^{n+1}}\,, \tag{5.37}$$

where, as in the n-winding case, $r_* = r(\theta=\pi/2)$.

An interesting observation concerning the solution we have just found is that, by choosing appropriately the winding number m, one of the spikes of the $m=0$ solutions at $\theta=0$ or $\theta=\pi$ disappears. Indeed if, for example, n is nonnegative and we take $2m=n+1$, the function $r(\theta)$ is regular at $\theta=0$. Similarly, also when $n\geq 0$, one can eliminate the spike at $\theta=\pi$ by choosing $2m=-n-1$.

5.3 Spiral solitons

By considering more general solutions of the Cauchy-Riemann equations (5.5) and (5.12) we can obtain many more classes of supersymmetric configurations of the brane probe. One of the questions one can address is whether or not one can have embeddings in which r is finite for all values of the angles. We will now see that the answer to this question is yes, although the corresponding embeddings seem not to be very interesting. To illustrate this point, let us see how we can find functions ψ and Λ such that they make the radial coordinate of the n-winding embedding finite at $\theta=0,\pi$. First of all, notice that, in terms of the Cauchy-Riemann variables u and $\tilde{u}$ defined in eq. (5.4), we have to explore the behaviour of the embedding at $u,\tilde{u}\rightarrow\pm\infty$. Since

$$\sin\theta = \frac{2e^u}{1+e^{2u}}\,, \qquad \sin\tilde{\theta} = \frac{2e^{\tilde{u}}}{1+e^{2\tilde{u}}}\,, \tag{5.38}$$

one has that $\sin\theta\rightarrow e^{-|u|}$, $\sin\tilde{\theta}\rightarrow e^{-|\tilde{u}|}$ as $u,\tilde{u}\rightarrow\pm\infty$. Then, the factors multiplying e^{Λ} in eq. (5.14) diverge as $e^{|u|+|\tilde{u}|}$. In the n-winding solution $|\tilde{u}|=|n||u|$ and, therefore this divergence is of the type $e^{(|n|+1)|u|}$. We can cancel this divergence by adding a Λ such that $e^{\Lambda}\rightarrow 0$ as $u\rightarrow\pm\infty$ in such a way that, for example, $\Lambda+(|n|+1)|u|\rightarrow-\infty$. This is clearly achieved by taking a function Λ such that $\Lambda\rightarrow -u^2$. It is straightforward to find an analytic function in the (u,φ) plane such that its imaginary part behaves as $-u^2$ for $u\rightarrow\pm\infty$. One can take

$$\psi + i\Lambda = -i(u+i\varphi)^2 = 2u\varphi - i(u^2-\varphi^2)\,. \tag{5.39}$$

From this equation we can read the functions ψ and Λ. In terms of θ and φ they are:

$$\psi = 2u\varphi = 2\varphi\log\tan\frac{\theta}{2}\,, \qquad \Lambda = -u^2+\varphi^2 = -\left(\log\tan\frac{\theta}{2}\right)^2 + \varphi^2\,. \tag{5.40}$$

In this case $r\rightarrow 0$, $\psi\rightarrow\pm\infty$ as $\theta\rightarrow 0,\pi$, which means that we describe an infinite spiral which winds infinitely in the ψ direction. Notice that, although r is always finite, the volume of the two-submanifold is infinite due to this infinite winding. One can try other alternatives to make the radial coordinate finite. In all the ones we have analyzed one obtains the infinite spiral behaviour described above.

JHEP12(2003)024

6. Worldvolume solitons (nonabelian case)

Let us consider the full nonabelian background and let us try to obtain solutions to the kappa symmetry equations (3.9)–(3.13). Actually we will restrict ourselves to the situations in which r only depends on the angle θ. It can be easily checked that, in this case, only four of the five equations (3.9)–(3.13) are independent. As an independent set of equations we will choose eqs. (3.9), (3.17) and

$$\partial_\theta r = -\frac{e^h\, V_{3\varphi}}{e^h\, \cos\alpha + (V_{2\varphi} - \frac{a}{2})\, \sin\alpha}\,, \tag{6.1}$$

$$\sin\alpha V_{1\varphi}\, \partial_\theta r - e^h\, V_{3\theta} = 0\,, \tag{6.2}$$

which can be obtained from eqs. (3.10) and (3.13) after taking $\partial_\varphi r = 0$.

We will now try to find the non-abelian version of the solutions found in the abelian theory for arbitrary winding n. With this purpose, let us consider the following ansatz for $\tilde{\varphi}$:

$$\tilde{\varphi}(\theta,\varphi) = n\varphi + f(\theta)\,, \tag{6.3}$$

while we shall assume that $\tilde{\theta}$, ψ and r are functions of θ only. We will require that, in the asymptotic UV, $\tilde{\varphi} \to n\varphi$. It this clear that in this ansatz $\partial_\varphi\tilde{\varphi} = n$ and that $\partial_\theta\tilde{\varphi} = \partial_\theta f$. Moreover, from eq. (3.9) we can obtain the relation between $\partial_\theta\tilde{\varphi}$ and $\partial_\theta\tilde{\theta}$, namely:

$$\partial_\theta\tilde{\varphi} = \tan\psi \left[\frac{\partial_\theta\tilde{\theta}}{\sin\tilde{\theta}} - \frac{n}{\sin\theta}\right]. \tag{6.4}$$

Using this value of $\partial_\theta\tilde{\varphi}$, we get the following values of the V functions:

$$\begin{aligned}
V_{1\theta} &= \frac{1}{2}\left[\frac{\partial_\theta\tilde{\theta}}{\cos\psi} - n\,\frac{\sin\tilde{\theta}}{\sin\theta}\,\frac{\sin^2\psi}{\cos\psi}\right],\\
V_{1\varphi} &= \frac{n}{2}\,\frac{\sin\tilde{\theta}}{\sin\theta}\,\sin\psi = -V_{2\theta}\,,\\
V_{2\varphi} &= \frac{n}{2}\,\frac{\sin\tilde{\theta}}{\sin\theta}\,\cos\psi\,,\\
V_{3\theta} &= \frac{1}{2}\,\partial_\theta\psi + \frac{1}{2}\,\cot\tilde{\theta}\,\tan\psi\,\partial_\theta\tilde{\theta} - \frac{n}{2}\,\tan\psi\,\frac{\cos\tilde{\theta}}{\sin\theta}\,,\\
V_{3\varphi} &= \frac{n}{2}\,\frac{\cos\tilde{\theta}}{\sin\theta} + \frac{1}{2}\,\cot\theta\,.
\end{aligned} \tag{6.5}$$

By using these values in eq. (3.17), one gets the value of $\partial_\theta\tilde{\theta}$ in terms of the other variables:

$$\partial_\theta\tilde{\theta} = \frac{n\sin\tilde{\theta}\cosh 2r - \sin\theta\cos\psi}{\sin\theta\cosh 2r - n\sin\tilde{\theta}\cos\psi}\,. \tag{6.6}$$

On the other hand, by combining the two equations (6.4) and (6.6), we obtain:

$$\partial_\theta\,\tilde{\varphi} = \frac{n^2\sin^2\tilde{\theta} - \sin^2\theta}{\sin\theta\sin\tilde{\theta}\,(\sin\theta\cosh 2r - n\sin\tilde{\theta}\cos\psi)}\,\sin\psi\,. \tag{6.7}$$

JHEP12(2003)024

Moreover, plugging the values of $V_{1\varphi}$, $V_{2\varphi}$, $V_{3\theta}$ and $V_{3\varphi}$ in eqs. (6.1) and (6.2) we can obtain the values of the derivatives of ψ and r. The result is:

$$\begin{aligned}
\partial_\theta \psi &= \frac{n \cot\theta \sin\tilde{\theta} + \cot\tilde{\theta}\sin\theta}{\sin\theta\cosh 2r - n\sin\tilde{\theta}\cos\psi} \sin\psi\,, \\
\partial_\theta r &= -\frac{1}{2}\frac{n\cos\tilde{\theta} + \cos\theta}{\sin\theta\cosh 2r - n\sin\tilde{\theta}\cos\psi}\sinh 2r\,.
\end{aligned} \tag{6.8}$$

It follows from eq. (6.6) that, asymptotically in the UV, $\sin\theta\,\partial_\theta\tilde{\theta} \to n\sin\tilde{\theta}$. In order to fulfill this relation for arbitrary r it is easy to see from eq. (6.6) that one must have $(n\sin\tilde{\theta})^2 = \sin^2\theta$, which only happens for $n = \pm 1$ and $\sin\theta = \sin\tilde{\theta}$. Noticing that for these values one has $\partial_\theta\tilde{\theta} = \pm 1$, one is finally led to the two possibilities of eq. (5.22): $\tilde{\theta} = \theta$ for $n = 1$ and $\tilde{\theta} = \pi - \theta$ for $n = -1$. Notice that in the two cases of eq. (5.22) this equation implies that $\partial_\theta\,\tilde{\varphi} = 0$ and thus when $n = \pm 1$ the angular identifications of the abelian unit-winding embeddings (eq. (5.22)) also solve the non-abelian equations (6.6) and (6.7) for all r.

For a general value of n one has that asymptotically in the UV $\partial_\theta\tilde{\varphi} \to 0$ and $\partial_\theta\psi \to 0$, as in the abelian solutions. Moreover it follows from eqs. (6.7) and (6.8) that $\tilde{\varphi}$ and ψ can be kept constant for all r if $\sin\psi = 0$, i.e. when $\psi = 0, \pi \mod 2\pi$. For this values of ψ the equations simplify and, although we will not attempt to do it here, one could try to integrate numerically the equations of $\tilde{\theta}$ and r. It is however interesting to point out that, contrary to what happens in the abelian n-winding solution, the angle ψ cannot be an arbitrary constant for the non-abelian probes. As we will argue below, this is a geometrical realization of the breaking of the R-symmetry of the corresponding $\mathcal{N} = 1$ SYM theory in the IR. On the contrary, the angle $\tilde{\varphi}$ can take an arbitrary constant value, as in the abelian solution.

6.1 Non abelian unit-winding solutions

Let us now obtain the non-abelian generalization of the unit-winding solutions. First of all we define

$$\eta = n = \pm 1\,. \tag{6.9}$$

We have already noticed that for unit-winding embeddings the values of $\tilde{\theta}$ and $\tilde{\varphi}$ displayed in eq. (5.22) solve the non-abelian differential equations (6.6) and (6.7). Therefore, let us try to find a solution in the nonabelian theory in which the embedding of the $(\tilde{\theta}, \tilde{\varphi})$ coordinates is the same as in the abelian theory, i.e. as in eq. (5.22). For this type of embeddings $\sin\tilde{\theta} = \sin\theta$, $\partial_\theta\tilde{\theta} = \eta$ and eq. (6.5) reduces to:

$$\begin{aligned}
V_{1\theta} &= \frac{\eta\cos\psi}{2}\,, & V_{1\varphi} &= \frac{\eta\sin\psi}{2}\,, \\
V_{2\theta} &= -\frac{\eta\sin\psi}{2}\,, & V_{2\varphi} &= \frac{\eta\cos\psi}{2}\,, \\
V_{3\theta} &= \frac{\psi_\theta}{2}\,, & V_{3\varphi} &= \cot\theta\,,
\end{aligned} \tag{6.10}$$

where we have denoted $\psi_\theta \equiv \partial_\theta \, \psi$. As a check, notice that $V_{1\theta}$, $V_{1\varphi}$, $V_{2\theta}$ and $V_{2\varphi}$ satisfy eqs. (5.1). It follows from eq. (3.17) that they must also satisfy

$$V_{1\theta}^2 + V_{1\varphi}^2 = \frac{1}{4}\,, \qquad V_{2\theta}^2 + V_{2\varphi}^2 = \frac{1}{4}\,, \tag{6.11}$$

which indeed they verify. Moreover, by substituting $\sin\tilde{\theta} = \sin\theta$ and $\cos\tilde{\theta} = \eta\cos\theta$ in eq. (6.8), we obtain the following differential equations for $\psi(\theta)$ and $r(\theta)$:

$$\begin{aligned} \psi_\theta &= -\frac{2\eta\sin\psi}{\sinh 2r}\, r_\theta\,, \\ r_\theta &= -\frac{\cot\theta}{\cosh 2r - \eta\cos\psi}\, \sinh 2r\,. \end{aligned} \tag{6.12}$$

These equations can be integrated with the result:

$$\begin{aligned} \Big(\tan\frac{\psi}{2}\Big)^{\eta} &= A\,\coth r\,, \\ \frac{\sinh r}{\sqrt{A^2 + \tanh^2 r}} &= \frac{C}{\sin\theta}\,, \end{aligned} \tag{6.13}$$

where A and C are constants of integration. Eq. (6.13), together with eq. (5.22), determines the unit-winding embeddings of the probe in the non-abelian background. Notice that, as in the corresponding abelian solution, r diverges when $\theta = 0, \pi$, i.e. the brane probe extends infinitely in the radial direction. On the other hand, it is also instructive to explore the $r \to \infty$ limit of the solution (6.13). First of all, it is clear that when $r \to \infty$ the angle ψ reaches asymptotically a constant value ψ_0, given by

$$\cos\psi_0 = \frac{1 - A^2}{1 + A^2}\,\eta\,. \tag{6.14}$$

Moreover, when $r \to \infty$ the function $r(\theta)$ displayed in eq. (6.13) becomes, after a proper identification of the integration constants, exactly the one written in eq. (5.19) for $n = \pm 1$ and $c = 1$. Notice that the angle ψ in the embedding (6.13) is not constant in general. Actually, only when $A = 0$ or $A = \infty$ the coordinate ψ remains constant and equal to $0, \pi \mod 2\pi$ ($\cos\psi = \eta$ for $A = 0$ and $\cos\psi = -\eta$ in the case $A = \infty$). It is interesting to write the dependence of r on θ in these two particular cases. When $A = \infty$ the solution is

$$\begin{aligned} \tilde{\theta} &= \begin{cases} \theta, & \text{if } \eta = +1, \\ \pi - \theta, & \text{if } \eta = -1, \end{cases} \qquad & \tilde{\varphi} &= \begin{cases} \varphi + \text{constant}, & \text{if } \eta = +1, \\ 2\pi - \varphi + \text{constant}, & \text{if } \eta = -1 \end{cases} \\ \psi &= \begin{cases} \pi, 3\pi, & \text{if } \eta = +1, \\ 0, 2\pi, & \text{if } \eta = -1, \end{cases} & \sinh r &= \frac{\sinh r_*}{\sin\theta}\,, \end{aligned} \tag{6.15}$$

where r_* is the minimal value of r (i.e. $r_* = r(\theta = \pi/2)$) and we have also displayed the angular part of the embedding. Notice that, for a given sign of the winding number η, only two values of ψ are possible. Thus, in this solution, the U(1) symmetry of shifts in ψ is broken to a $\mathbb{Z}_2$ symmetry. This will be interpreted in section 7 as the realization, at the level of the brane probe, of the R-symmetry breaking of the gauge theory.

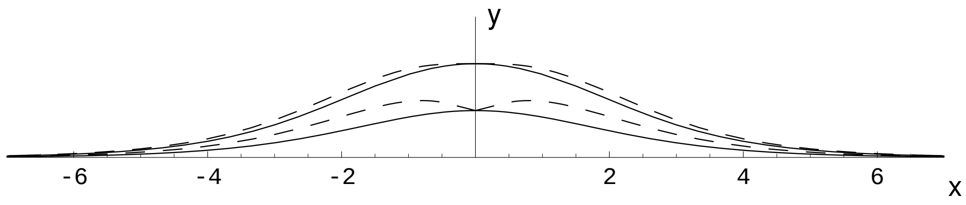

Figure 2: Comparison between the non-abelian (solid line) and abelian (dashed line) unit-winding embeddings for the same value of r_*. The non-abelian embedding is the one corresponding to eq. (6.15) and the abelian one is that given in eq. (5.19) for $n = 1$ and $c = 1$. The curves for two different values of r_* ($r_* = 0.5$ and $r_* = 1$) are shown. The variables (x, y) are the ones defined in eq. (5.20).

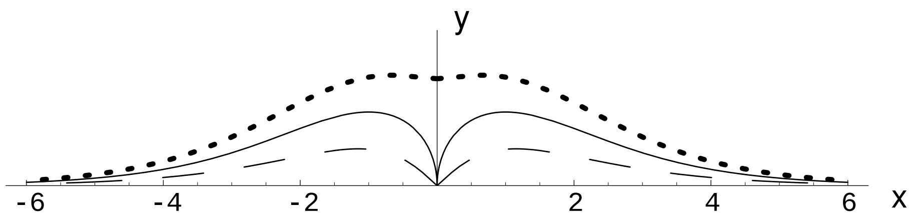

Figure 3: Graphic representation of the unit winding embedding of eq. (6.16) for three values of the constant C: $C = 0.5$ (dashed line), $C = 1$ (solid line) and $C = 1.5$ (dotted line). The variables (x, y) are the ones defined in eq. (5.20).

To have a better understanding of the solution (6.15) we have plotted it in figure 2 in terms of the variables (x, y) defined in eq. (5.20). For comparison we have also plotted the abelian solution corresponding to the same value of r_*. In this figure the embeddings for two different values of the minimal radial distance r_* are shown. When r_* is large enough ($r_* \geq 2$) the two curves become practically identical.

Let us now have a look at the case of the $A = 0$ embeddings. The function $r(\theta)$ in this case can be read from eq. (6.13), namely:

$$\cosh r = \frac{C}{\sin\theta}\,. \tag{6.16}$$

We have plotted in figure 3 the profile for these embeddings in terms of the variables (x, y) of eq. (5.20). Notice that, when C is in the interval $(1, \infty)$ it can be parametrized as $C = \cosh r_*$, with $r_* > 0$ being the minimal radial distance between the probe and the origin. On the contrary, when C lies in the interval $[0, 1]$ the brane reaches the origin when $\sin\theta = C$. We have thus, in this case, a one-parameter family of configurations which pass through the origin.

As in their abelian counterparts, these worldvolume solitons for the non-abelian background saturate an energy bound. In order to prove this fact, let us define

$$D \equiv \coth 2r - \eta\,\frac{\cos\psi}{\sinh 2r}\,. \tag{6.17}$$

Notice that $D \geq 0$ for any real ψ and r. Moreover the equation for $r(\theta)$ can be written as $r_\theta = -\cot\theta/D$. For arbitrary functions $r(\theta)$ and $\psi(\theta)$ the hamiltonian density takes the form:

$$\mathcal{H} = e^{2\phi}\sin\theta\Bigg[\sqrt{(r - r_\theta\cot\theta)^2 + rD\left(r_\theta + \frac{\cot\theta}{D}\right)^2 + \frac{rD}{4}\left(\psi_\theta - \frac{2\eta\sin\psi}{D\sinh 2r}\cot\theta\right)^2} - \\ - \left(2e^{2h} - \frac{1}{8}\left(a^2-1\right)^2 e^{-2h}\right)\cot\theta\, r_\theta\Bigg]\,. \tag{6.18}$$

It can be verified that $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{Z} + \mathcal{S}$, where

$$\mathcal{Z} = -\frac{d}{d\theta}\left[e^{2\phi}\, r\cos\theta\right]. \tag{6.19}$$

(this is the same value as in the abelian soliton for $n = 1$). The expression of $\mathcal{S}$ is:

$$\mathcal{S} = e^{2\phi}\sin\theta\Bigg[\sqrt{(r - r_\theta\cot\theta)^2 + rD\left(r_\theta + \frac{\cot\theta}{D}\right)^2 + \frac{rD}{4}\left(\psi_\theta - \frac{2\eta\sin\psi}{D\sinh 2r}\cot\theta\right)^2} - \\ - (r - r_\theta\cot\theta)\Bigg]\,. \tag{6.20}$$

As in the abelian case, if the BPS equations (6.12) are satisfied, the square root on eq. (6.20) can be exactly evaluated and $\mathcal{S}$ vanishes. Furthermore, one can easily check that $\mathcal{S} \geq 0$ is equivalent to the condition

$$rD\left(r_\theta + \frac{\cot\theta}{D}\right)^2 + \frac{rD}{4}\left(\psi_\theta - \frac{2\eta\sin\psi}{D\sinh 2r}\cot\theta\right)^2 \geq 0\,, \tag{6.21}$$

which, since $D \geq 0$, is trivially satisfied for any functions $r(\theta)$ and $\psi(\theta)$. Moreover, as $r - r_\theta\cot\theta \geq 0$ for the solution of the BPS equations, it follows that our BPS embedding saturates the bound.

6.2 Non abelian zero-winding solutions

The differential equations for the non-abelian version of the zero-winding solution can be obtained by putting $n = 0$ in our general equations. Actually, by taking $n = 0$ in the second equation in (6.8) one obtains the differential equation which determines the dependence of r on the angle θ, namely:

$$r_\theta = -\frac{\cot\theta}{2\coth 2r}\,, \tag{6.22}$$

which can be easily integrated, namely:

$$\sinh 2r = \frac{C}{\sin\theta}\,. \tag{6.23}$$

Notice that, as in the abelian case, this solution has two spikes at $\theta = 0, \pi$, where r diverges and, thus, the brane probe also extends infinitely in the radial direction. Moreover, the minimal value of the radial coordinate, which we will denote by r_*, is reached at $\theta = \pi/2$.

JHEP12(2003)024

This minimal value is related to the constant C in eq. (6.23), namely $\sinh 2r_* = C$. It is readily verified that for large r this solution behaves exactly as the zero-winding solution in the abelian theory. Moreover, it follows from eq. (6.4) that, in this $n = 0$ case, the angle $\tilde{\varphi}$ only depends on θ. Actually, the differential equations for the angles $\tilde{\theta}$, $\tilde{\varphi}$ and ψ as functions of θ are easily obtained from eqs. (6.6), (6.7) and (6.4):

$$\begin{aligned}
\partial_\theta\,\tilde{\theta} &= -\frac{\cos\psi}{\cosh 2r}\,,\\
\partial_\theta\,\tilde{\varphi} &= -\frac{1}{\cosh 2r}\,\frac{\sin\psi}{\sin\tilde{\theta}}\,,\\
\partial_\theta\,\psi &= \frac{\cot\tilde{\theta}\sin\psi}{\cosh 2r}\,.
\end{aligned} \tag{6.24}$$

By combining the equations of ψ and $\tilde{\theta}$ one can easily get the relation between these two angles, namely:

$$\sin\psi = \frac{B}{\sin\tilde{\theta}}\,. \tag{6.25}$$

Notice that, for consistency, $B \leq 1$ and $\sin\tilde{\theta} \geq B$. We can also obtain $\tilde{\varphi} = \tilde{\varphi}(\tilde{\theta})$ and $\tilde{\theta} = \tilde{\theta}(\theta)$:

$$\begin{aligned}
\tilde{\varphi} &= -\arctan\left[\frac{B\cos\tilde{\theta}}{\sqrt{\sin^2\tilde{\theta} - B^2}}\right] + \text{constant}\,,\\
-\arcsin\left[\frac{\cos\tilde{\theta}}{\sqrt{1-B^2}}\right] &= \arcsin\left[\frac{\cos\theta}{\sqrt{1+C^2}}\right] + \text{constant}\,.
\end{aligned} \tag{6.26}$$

Actually, much simpler equations for the embedding are obtained if one considers the particular case in which the angle ψ is constant. Notice that, as was pointed out after eq. (6.8), this only can happen if $\psi = 0, \pi \pmod{2\pi}$ (see also the last equation in (6.24)). These solutions correspond to taking the constant B equal to zero in eq. (6.25). Moreover, it follows from the eq. (6.24) that $\tilde{\varphi}$ is an arbitrary constant in this case, while the dependence of $\tilde{\theta}$ on θ can be obtained by combining eqs. (6.23) and (6.24). If we denote $\cos\psi = \epsilon$, with $\epsilon = \pm 1$, one has:

$$\sinh 2r = \frac{\sinh 2r_*}{\sin\theta}\,, \qquad \sin(\tilde{\theta} - \tilde{\theta}_*) = \epsilon\,\frac{\cos\theta}{\cosh 2r_*}\,, \tag{6.27}$$

where $\tilde{\theta}_* = \tilde{\theta}(\theta = \pi/2)$. Notice that there are four possible values of ψ in this zero-winding solution and, thus, the U(1) R-symmetry is broken to $\mathbb{Z}_4$ in this case.

Let us finally point out that, also in this case, the hamiltonian density $\mathcal{H}$ can be put as $\mathcal{H} = \mathcal{Z} + \mathcal{S}$, where $\mathcal{S}$ is nonnegative ($\mathcal{S} = 0$ for the BPS solution) and $\mathcal{Z}$ is given by:

$$\mathcal{Z} = -\partial_\theta\left[e^{2\phi}\cos\theta\left(r - \frac{1}{4}\coth 2r + \frac{r}{2\sinh^2 2r}\right)\right]. \tag{6.28}$$

6.3 Cylinder solutions

We shall now show that there exists a general class of supersymmetric embeddings for the non-abelian background. For convenience, let us consider r as worldvolume coordinate and let us assume that the D5-brane is sitting at the north poles of the (θ, φ) and $(\tilde{\theta}, \tilde{\varphi})$ two-spheres, i.e. at $\theta = \tilde{\theta} = 0$. In the remaining angular coordinates φ, $\tilde{\varphi}$ and ψ, the embedding is characterized by the equation:

$$\frac{\varphi - \varphi_0}{p} = \frac{\tilde{\varphi} - \tilde{\varphi}_0}{q} = \frac{\psi - \psi_0}{s}\,, \tag{6.29}$$

where $(\varphi_0, \tilde{\varphi}_0, \psi_0)$ and (p, q, s) are constants. Notice that, if one of the constants of the denominator in (6.29) is zero, then the corresponding angle must be a constant. Let us parametrize these embeddings by means of two worldvolume coordinates σ_1 and σ_2, defined as follows:

$$\sigma_1 = \frac{\varphi - \varphi_0}{p} = \frac{\tilde{\varphi} - \tilde{\varphi}_0}{q} = \frac{\psi - \psi_0}{s}\,, \qquad \sigma_2 = r\,. \tag{6.30}$$

It is straightforward to demonstrate that the pullback to the worldvolume of the forms w^i and A^i is given by:

$$\begin{aligned} &P[\,w^1\,] = P[\,w^2\,] = 0\,, \qquad && P[\,w^3\,] = (q+s)\,d\sigma_1\,, \\ &P[\,A^1\,] = P[\,A^2\,] = 0\,, \qquad && P[\,A^3\,] = -p\,d\sigma_1\,. \end{aligned} \tag{6.31}$$

It follows from these results that the pullback of the frame one-forms e^i and $e^{\hat{j}}$ is zero for $i, j = 1, 2$, whereas $P[e^{\hat{3}}]$ is non-vanishing. As a consequence, the induced Dirac matrices are:

$$\gamma_{\sigma_1} = \frac{1}{2}\,(p+q+s)\,e^{\frac{\phi}{4}}\,\hat{\Gamma}_3\,, \qquad \gamma_{\sigma_2} = e^{\frac{\phi}{4}}\,\Gamma_r\,. \tag{6.32}$$

The kappa symmetry matrix Γ_κ for the embedding at hand is:

$$\Gamma_\kappa = \frac{e^{\phi}}{\sqrt{-g}}\,\Gamma_{x^0\cdots x^3}\,\gamma_{\sigma_1\sigma_2}\,. \tag{6.33}$$

Moreover, by using the projection conditions satisfied by the Killing spinors ϵ, one can prove that

$$\gamma_{\sigma_1\sigma_2}\,\epsilon = -\frac{p+q+s}{2}\,e^{\frac{\phi}{2}}\,\Gamma_r\,\hat{\Gamma}_3\,\epsilon = \frac{p+q+s}{2}\,e^{\frac{\phi}{2}}\,(\cos\alpha\,\Gamma_{12} + \sin\alpha\Gamma_1\hat{\Gamma}_2)\,\epsilon\,, \tag{6.34}$$

and, since the determinant of the induced metric is $\sqrt{-g} = e^{\frac{3\phi}{2}}\,\frac{p+q+s}{2}$, it is immediate to verify that the kappa symmetry projection $\Gamma_\kappa\epsilon = \epsilon$ coincides with the projection satisfied by the Killing spinors of the background. Therefore, our brane probe preserves all the supersymmetries of the background. Notice that the induced metric on the worldvolume along the σ_1, σ_2 directions has the form

$$e^{\frac{\phi}{2}}\left[\frac{(p+q+s)^2}{4}\,d\sigma_1^2 + d\sigma_2^2\right], \tag{6.35}$$

which is conformally equivalent to the metric of a cylinder. After a simple calculation one can prove that the energy density of these solutions is

$$\mathcal{H} = \partial_r \left[e^{2\phi} \left(pr + (q+s-p) \left(\frac{\coth 2r}{4} - \frac{r}{2\sinh^2 2r} \right) \right) \right] . \tag{6.36}$$

One can also have cylinders located at the south pole of the (θ, φ) and $(\tilde{\theta}, \tilde{\varphi})$ two-spheres. Indeed, the above equations remain valid if $\theta = \pi$ ($\tilde{\theta} = \pi$) if one changes $p \to -p$ ($q \to -q$, respectively). On the other hand, if $p = 0$ the angle φ is constant and, as the pullback of the e^i frame one-forms also vanishes when θ is also constant, it follows that θ can have any constant value when $p = 0$. Similarly, if $q = 0$ one necessarily has $\tilde{\varphi} = \tilde{\varphi}_0$ and $\tilde{\theta}$ can be an arbitrary constant in this case.

When $p = 1$, $q = n$ and $s = 2m$, the angular part of the embedding is the same as in the (n, m)-winding solitons. Actually, these cylinder solutions correspond to formally taking $r_* \to -\infty$ is the abelian solution of eq. (5.37). This forces one to take $\theta = 0, \pi$ and, thus one can regard the cylinder as a zoom which magnifies the region in which the probe goes to infinity. One can also get cylinder embeddings by consider the limit of the non-abelian solutions in which the probe reaches the point $r = 0$. For example, by taking $r_* = 0$ in eq. (6.15) one gets the $p = q = 1$, $s = 0$ cylinder solutions while the $r_* \to 0$ limit of the embedding (6.27) corresponds to a cylinder with $p = 1$ and $q = s = 0$. Actually, when one takes the $r_* \to 0$ limit of these non-abelian embeddings one obtains two cylinder solutions, one with $\theta = 0$ and the other with $\theta = \pi$. This suggests that, in order to obtain a consistent solution, one must combine in general two cylinders located at each of the two poles of the (θ, φ) two-sphere. Notice that this is also required if one imposes the condition of RR charge neutrality of the two-sphere at infinity.

7. Quadratic fluctuations around the unit-winding embedding

As mentioned in the introduction, we are now going to consider some of the brane probe configurations previously found as flavor branes, which will allow us to introduce dynamical quarks in the $\mathcal{N} = 1$ SYM theory. Following refs. [19, 20, 21], the spectrum of quadratic fluctuations of the brane probe will be interpreted as the meson spectrum of $\mathcal{N} = 1$ SQCD. So, let us try to elaborate on the reasons to consider these probes as the addition of flavors to the field theory dual. In fact, when considering the 't Hooft expansion for large number of colors, the role of flavors is played by the boundaries of the Feynman graph. From a gravity perspective, these boundaries correspond to the addition of D-branes and open strings in the game.

In our case, we have a system of N D5-branes wrapping a two-cycle inside the resolved conifold and N_f D5-branes that wrap another two-submanifold, thus introducing N_f flavors in the SU(N) gauge theory. Taking the decoupling limit with $g_s \alpha' N$ fixed and large is equivalent to replacing the N D5-branes by the geometry they generate (the one studied in section 2), while the N_f-D5 branes that do not backreact (because we take N_f much smaller than N) are treated as probes. From a gauge theory perspective, this is equivalent to consider the dynamics of gluons and gluinos coupled to fundamentals, but neglecting the backreaction of the latter. Of course it would be of great interest to find the backreacted solution.

The way of adding fundamental fields in this gauge theory from a string theory perspective was discussed in [28], where two possible ways, adding $D9$ branes or adding $D5$ branes as probes, were proposed. In this paper we are considering the cleaner case of $D5$ probes. A careful analysis of the open string spectrum shows the existence of a four dimensional gauge $\mathcal{N}=1$ vector multiplet and a complex scalar multiplet. This is the spectrum of SQCD. In the case analyzed below we will consider abelian DBI actions for the probes, so that we will be dealing with the $N_f = 1$ case.

We have found several brane configurations in the non-abelian background which, in principle, could be suitable to generate the meson spectrum. One of the requirements we should demand to these configurations is that they must incorporate some scale parameter which could be used to generate the mass scale of the quarks. Within our framework such a mass scale is nothing but the minimal distance between the flavor brane and the origin, i.e. what we have denoted by r_*. This requirement allows to discard the cylinder solutions we have found since they reach the origin and have no such a mass scale. We are thus left with the unit-winding solutions of section 6.1 and the zero-winding solutions of section 6.2 as the only analytical solutions we have found for the non-abelian background.

In this section we shall analyze the fluctuations around the unit-winding solutions of section 6.1. We have several reasons for this election. First of all, the unperturbed unit-winding embedding is simpler. Secondly, we will show in appendix B that the UV behaviour of the fluctuations is better in the unit-winding configuration than in the zero-winding embedding. Thirdly, the unit-winding embeddings of constant ψ incorporate the correct pattern $\mathrm{U}(1) \to \mathbb{Z}_2$ of R-symmetry breaking, whereas for the zero-winding embeddings of eq. (6.27) the $\mathrm{U}(1)$ symmetry is broken to a $\mathbb{Z}_4$ subgroup.

Recall from section 6.1 that we have two possible solutions with $\psi = 0, \pi \pmod{2\pi}$, which are the ones displayed in eqs. (6.15) and (6.16). As discussed in section 6.1, the solution of eq. (6.16) contains a one-parameter subfamily of embeddings which reach the origin and, thus, they should correspond to massless dynamical quarks. On the contrary, the embeddings of eq. (6.15) pass through the origin only in one case, i.e. when $r_* = 0$ and, somehow, the limit in which the quarks are massless is uniquely defined. Recall that for $r_* = 0$ the solution (6.15) is identical to the unit-winding cylinder. For these reasons we consider the configuration displayed in eq. (6.15) more adequate for our purposes and we will use it as the unperturbed flavor brane.

We will consider first in subsection 7.1 the fluctuations of the scalar transverse to the transverse probe, while in subsection 7.2 we will study the fluctuations of the worldvolume gauge field. The gauge theory interpretation of the results will be discussed in subsection 7.3

7.1 Scalar mesons

Let us consider a non-abelian unit-winding embedding with $\tilde{\theta} = \theta$, $\tilde{\varphi} = \varphi +$ constant and $\psi = \pi \pmod{2\pi}$. For convenience we take first r as worldvolume coordinate and consider θ as a function of r, φ and of the unwrapped coordinates x, i.e. $\theta = \theta(r, x, \varphi)$. The lagrangian density for such embedding can be easily obtained by computing the induced metric. One gets:

$$\begin{aligned}\mathcal{L} &= -e^{2\phi}\sin\theta\times\\&\times\Bigg[\sqrt{\Big[1+r\tanh r\Big((\partial_r\theta)^2+(\partial_x\theta)^2+\frac{1}{\cos^2\theta+r\coth r\sin^2\theta}(\partial_\varphi\theta)^2\Big)\Big](r\coth r+\cot^2\theta)}+\\&\qquad+r\partial_r\theta-\cot\theta\Bigg],\end{aligned} \tag{7.1}$$

where we have neglected the term $\partial_r(r\cos\theta e^{2\phi})$ which, being a total radial derivative, does not contribute to the equations of motion.

We are going to expand this lagrangian around the corresponding non-abelian unit-winding configuration obtained in section 6.1. Actually, by taking in eq. (6.15) $\eta = +1$ one obtains a configuration with $\psi = \pi \pmod{2\pi}$, which corresponds to a function $\theta = \theta_0(r)$, given by:

$$\sin\theta_0(r) = \frac{\sinh r_*}{\sinh r}, \tag{7.2}$$

where r_* is the minimum value of r and $r_* \leq r < \infty$. It is clear from this equation that with the coordinate r we can only describe one-half of the brane probe: the one that is wrapped, say, on the north hemisphere of the two-sphere, in which $\theta \in (0, \pi/2)$. Outside this interval $\theta_0(r)$ is a double-valued function of r. Let us put

$$\theta(r,x,\varphi) = \theta_0(r) + \chi(r,x,\varphi), \tag{7.3}$$

and expand $\mathcal{L}$ up to quadratic order in χ. Using the first-order equation satisfied by $\theta_0(r)$, namely:

$$\partial_r\theta_0 = -\coth r\tan\theta_0, \tag{7.4}$$

we get

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\frac{e^{2\phi}}{1+r\coth r\tan^2\theta_0}\left[r\tanh r\cos\theta_0\,(\partial_r\chi)^2+\frac{2r}{\cos\theta_0}\chi\,\partial_r\chi+\frac{r\coth r}{\cos^3\theta_0}\chi^2\right]-\\&\quad-\frac{1}{2}e^{2\phi}\,r\tanh r\left[\cos\theta_0\,(\partial_x\chi)^2+\frac{1}{\cos\theta_0\,(1+r\coth r\tan^2\theta_0)}(\partial_\varphi\chi)^2\right].\end{aligned} \tag{7.5}$$

In the equations of motion derived from this lagrangian we will perform the ansatz:

$$\chi(r,x,\varphi) = e^{ikx}\,e^{il\varphi}\xi(r), \tag{7.6}$$

where, as φ is a periodically identified coordinate, l must be an integer and k is a four-vector whose square determines the four-dimensional mass M of the fluctuation mode:

$$M^2 = -k^2. \tag{7.7}$$

By substituting the functions (7.6) in the equation of motion that follows from the lagrangian (7.5), one gets a second order differential equation which is rather complicated and that can only be solved numerically. However, it is not difficult to obtain analytically the asymptotic behaviour of $\xi(r)$. This has been done in appendix B and we will now use these results to explore the nature of the fluctuations. For large r, i.e. in the UV, one gets (see eq. (B.12)) that $\xi(r)$ vanishes exponentially in the form:

$$\xi(r) \sim \frac{e^{-r}}{r^{\frac{1}{4}}} \cos\left[\sqrt{M^2 - l^2}\, r + \delta\right], \qquad (r \to \infty)\,, \tag{7.8}$$

where δ is a phase and we are assuming that $M^2 \geq l^2$. For $M^2 < l^2$ the fluctuations do not oscillate in the UV and we will not be able to impose the appropriate boundary conditions (see below). Notice that our unperturbed solution $\theta_0(r)$ also decreases in the UV as

$$\theta_0(r) \sim e^{-r} \qquad (r \to \infty)\,. \tag{7.9}$$

Thus $\xi(r)/\theta_0(r) \to 0$ as $r \to \infty$ and the first-order expansion we are performing continues[2] to be valid in the UV. On the other hand, for r close to r_* there are two independent solutions, one of them is finite at $r = r_*$ while the other diverges as

$$\xi(r) \sim \frac{1}{\sqrt{r - r_*}}\,. \tag{7.10}$$

Let us now see how one can use the information on the asymptotic behaviour of the fluctuation modes to extract their value for the full range of the radial coordinate. First of all, it is clear that, in principle, by consistency with the type of expansion we are adopting, one should require that $\xi << \theta_0$. Thus, one should discard the solutions which diverge in the infrared (see, however, the discussion below). Moreover, the behaviour of the fluctuations ξ for large r should be determined by some normalizability conditions. The corresponding norm would be an expression of the form:

$$\int_0^\infty dr \sqrt{\gamma}\xi^2\,, \tag{7.11}$$

where $\sqrt{\gamma}$ is some measure, which can be determined by looking at the lagrangian (7.5). If we regard χ as a scalar field with the standard normalization in a curved space, then $\sqrt{\gamma}$ is just the coefficient of the kinetic term $\frac{1}{2}(\partial_r\chi)^2$ in $\mathcal{L}$, namely

$$\sqrt{\gamma} = e^{2\phi} \frac{r \tanh r \cos\theta_0}{1 + r \coth r \tan^2\theta_0}\,. \tag{7.12}$$

For large r, $\sqrt{\gamma}$ behaves as

$$\sqrt{\gamma}(r \to \infty) \approx r^{\frac{1}{2}}\, e^{2r}\,. \tag{7.13}$$

Notice that the factors on the right-hand side of eq. (7.13) cancel against the exponentials and power factors of ξ^2 in the UV (see eq. (7.8)). As a consequence, all solutions have infinite norm.

[2]For the zero-winding solution, on the contrary, the ratio $\xi(r)/\theta_0(r)$ diverges in the UV (see appendix B).

The reason for the bad behaviour we have just discovered is the exponential blow up of the dilaton in the UV which invalidates the supergravity approximation. Actually, if one wishes to push the theory to the UV one has to perform an S-duality, which basically changes $e^{2\phi}$ by $e^{-2\phi}$. The S-dual theory corresponds to wrapped Neveu-Schwarz fivebranes and is the supergravity dual of a little string theory. Notice that, by changing $e^{2\phi} \to e^{-2\phi}$ in the measure (7.12), all solutions become normalizable, which is as bad as having no normalizable solutions at all. Moreover, it is unclear how to perform an S-duality in our D5-brane probe and convert it in a Neveu-Schwarz fivebrane for large values of the radial coordinate.

A problem similar to the one we are facing here appeared in ref. [30] in the calculation of the glueball spectrum for this background. It was argued in this reference that, in order to have a discrete spectrum, one has to introduce a cut-off to discriminate between the two regimes of the theory. Notice that, since they extend infinitely in the radial direction, we cannot avoid that our D5-brane probe explores the deep UV region. However, what we can do is to consider fluctuations that are significantly nonzero only on scales in which one can safely trust the supergravity approximation. In ref. [30] it was proposed to implement this condition by requiring the fluctuation to vanish at some conveniently chosen UV cut-off Λ. Translated to our situation, this proposal amounts to requiring:

$$\xi(r)|_{r=\Lambda} = 0\,. \tag{7.14}$$

This condition, together with the regularity of $\xi(r)$ at $r = r_*$, produces a discrete spectrum which we shall explore below. Notice that, for consistency with the general picture described above, in addition to having a node at $r = \Lambda$ as in eq. (7.14), the function $\xi(r)$ should be small for r close to the UV cut-off. This condition can be fulfilled by adjusting appropriately the mass scale of our solution, i.e. the minimal distance r_* between the probe and the origin, in such a way that r_* is not too close to Λ.

Notice that, by imposing the boundary condition (7.14) on the fluctuations, we are effectively introducing an infinite wall located at the UV cut-off. The introduction of this wall allows to have a discrete spectrum and should be regarded as a physical condition which implements the correct range of validity of the background geometry as a supergravity dual of $\mathcal{N} = 1$ Yang-Mills. Even if this regularisation could appear too rude and unnatural, the results obtained by using it for the first glueball masses are not too bad [30].

The cut-off scale Λ should not be a new scale but instead it should be obtainable from the background geometry itself. The proposal of ref. [30] is to take Λ as the scale of gaugino condensation, which is believed to correspond to the point at which the function $a(r)$ approaches its asymptotic value $a = 0$. A more pragmatic point of view, to which we will adhere here, is just taking the value of Λ which gives reasonable values for the glueball masses. In ref. [30] the value $\Lambda = 2$ was needed to fit the glueball masses obtained from lattice calculations, whereas with $\Lambda = 3.5$ one gets a glueball spectrum which resembles that predicted by other supergravity models. Notice that from $r = 0$ to $r = 3$ the effective string coupling constant e^{ϕ} increases in an order of magnitude. From our point of view it is also natural to look at the effect of the background on our brane probe. In this sense it is interesting to point out that for $r_* = 2 - 3$ onwards the abelian and non-abelian embeddings are indistinguishable (see sect. 6.1).

JHEP12(2003)024

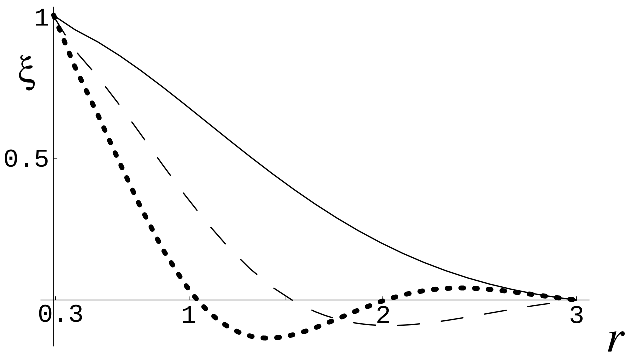

Figure 4: Graphic representation of the first three fluctuation modes for $r_* = 0.3$, $\Lambda = 3$ and $l = 0$. The three curves have been normalized to have $\xi(r_*) = 1$.

We have performed the numerical integration of the equation of motion of $\xi(r)$ subject to the boundary condition (7.14) by means of the shooting technique. For a generic value of the mass M the solution diverges at $r = r_*$. Only for some discrete set of masses the fluctuations are regular in the IR. In figure 4 we show the first three modes obtained by this procedure for $r_* = 0.3$ and $\Lambda = 3$. From this figure, we notice that the number of zeroes of $\xi(r)$ grows with the mass. In general one observes that the n^{th} mode has $n-1$ nodes in the region $r_* < r < \Lambda$, in agreement with the general expectation for this type of boundary value problems. Moreover, for $l = 0$, the mass M grows linearly with the number of nodes (see below).

At this point it is interesting to pause a while and discuss the suitability of our election of r as worldvolume coordinate. Although this coordinate is certainly very useful to extract the asymptotic behavior of the fluctuations (specially in the UV), we should keep in mind that we are only describing one half of the brane, i.e. the one corresponding to one of the two hemispheres of the two-sphere. On the other hand, the election of the angle as excited scalar has some subtleties which we now discuss. Actually, to describe the displacement of the brane probe with respect to its unperturbed configuration it is physically more sensible to use the coordinate y, defined in eq. (5.20). Accordingly, let us define the function $y(r, x, \varphi)$ as

$$y(r, x, \varphi) = r \sin\theta(r, x, \varphi)\,. \tag{7.15}$$

Let us put in this equation $\theta(r, x, \varphi) = \theta_0(r) + \chi(r, x, \varphi)$. At the linear order in χ we are working, y can be written as

$$y(r, x, \varphi) = y_0(r) + r\cos\theta_0(r)\,\chi(r, x, \varphi)\,, \tag{7.16}$$

where $y_0(r) \equiv r\sin\theta_0(r)$ corresponds to the unperturbed brane. Notice, first of all, that for those modes in which $\chi(r_*, x, \varphi)$ is finite, the fluctuation term in $y(r_*, x, \varphi)$ vanishes since $\cos\theta_0(r) \to 0$ when $r \to r_*$. Then

$$y(r_*, x, \varphi) = y_0(r_*) = r_* \qquad \text{if } \chi(r_*, x, \varphi) \qquad \text{is finite}\,. \tag{7.17}$$

Thus, by considering those modes χ that are regular at $r = r_*$ we are effectively restricting ourselves to the modes which have a node in the y coordinate at $r = r_*$. If, on the contrary χ diverges for $r \approx r_*$, we know from eq. (7.10) that it behaves as $\chi \approx 1/\sqrt{r - r_*}$. But we also know that $\cos\theta_0(r) \to 0$ when $r \to r_*$ and, in fact, the second term on the right-hand side of eq. (7.16) remains undetermined. The precise form in which $\cos\theta_0(r_*)$ vanishes can be read from eq. (B.13), namely $\cos\theta_0(r) \approx \sqrt{r - r_*}$ for $r \approx r_*$. Therefore, even if χ diverges at $r = r_*$, we could have, in the linearized approximation we are adopting, a finite value for $y(r_*, x, \varphi)$. Thus, modes with $\chi(r_*, x, \varphi) \to \infty$ should not be discarded. Actually, $y(r_*, x, \varphi) - y_0(r_*)$, although finite, is undetermined in eq. (7.16) for these modes and, in order to obtain its allowed values we should impose a boundary condition at the other half of the brane. As previously mentioned, this cannot be done when r is taken as worldvolume coordinate. Therefore, it is convenient to come back to the formalism of sects. 3-5, in which θ had been chosen as one of the worldvolume coordinates. In this approach, the unperturbed brane configuration is described by a function $r_0(\theta)$, given by:

$$\sinh r_0(\theta) = \frac{\sinh r_*}{\sin\theta}\,, \tag{7.18}$$

and the brane embedding is characterized by a function $r = r(\theta, x, \varphi)$, which we expand around $r_0(\theta)$ as follows:

$$r(\theta, x, \varphi) = r_0(\theta) + \rho(\theta, x, \varphi)\,. \tag{7.19}$$

Plugging this expansion into the DBI lagrangian of eq. (5.24) and keeping up to second order terms in ρ, one gets the following lagrangian density:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\frac{e^{2\phi}\sin\theta r_0}{r_0 + \cot^2\theta\tanh r_0}\left[\coth r_0(\partial_\theta\rho)^2 + \frac{2\cot\theta}{\sinh r_0\cosh r_0}\rho\partial_\theta\rho + \frac{\cot^2\theta}{\sinh r_0\cosh^3 r_0}\rho^2\right] - \\ &-\frac{e^{2\phi}\sin\theta r_0}{2}\left[(\partial_x\rho)^2 + \frac{1}{\cos^2\theta(1 + r_0\coth r_0\tan^2\theta)}(\partial_\varphi\rho)^2\right]\,. \end{aligned} \tag{7.20}$$

Similarly to what we have done with the lagrangian (7.5), we will look for solutions of the equations of motion of $\mathcal{L}$ which have the form:

$$\rho(\theta, x, \varphi) = e^{ikx}\, e^{il\varphi}\zeta(\theta)\,, \tag{7.21}$$

where l is an integer and $k^2 = -M^2$. As before, in order to get a discrete spectrum one must impose some boundary conditions. In the present approach we should cutoff the regions close to the two poles of the two-sphere. Accordingly, let us define the following angle

$$\sin\theta_\Lambda \equiv \frac{\sinh r_*}{\sinh\Lambda}\,, \qquad \theta_\Lambda \in \left(0, \frac{\pi}{2}\right)\,, \tag{7.22}$$

where, as indicated, we are taking θ_Λ in the range $0 < \theta_\Lambda < \pi/2$. Notice that $\theta_\Lambda \to 0$ if $\Lambda \to \infty$ as it should. Clearly θ_Λ and $\pi - \theta_\Lambda$ are the two angles that correspond to the radial scale Λ. Therefore, we impose the following boundary conditions to our fluctuation

$$\zeta(\theta_\Lambda) = \zeta(\pi - \theta_\Lambda) = 0\,. \tag{7.23}$$

JHEP12(2003)024

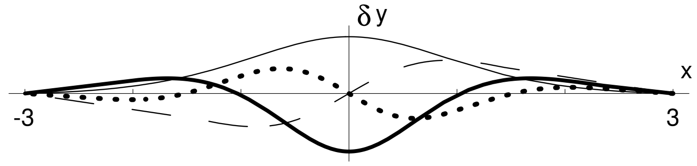

Figure 5: Plot of $\delta y \equiv \zeta \sin\theta$ versus $x = r\cos\theta$ for the first four modes for $r_* = 0.3$, $\Lambda = 3$ and $l = 0$. The dashed and dotted curves pass through the origin and correspond to the first two modes of figure 3.

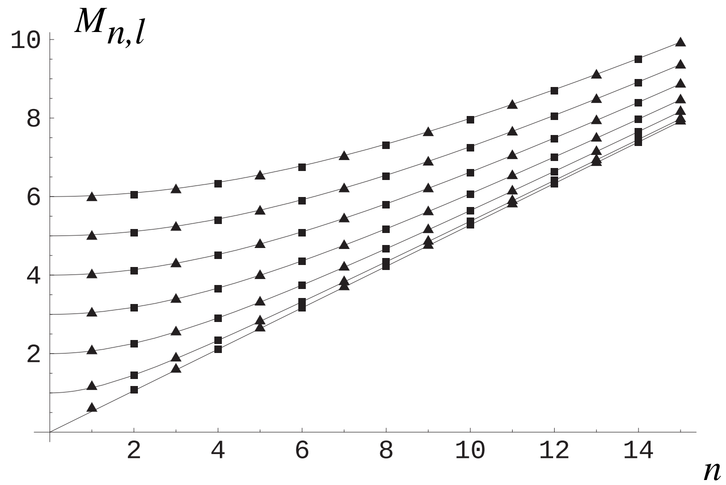

Figure 6: Mass spectrum for $r_* = 0.3$ and $\Lambda = 3$ for $l = 0, \cdots, 6$. The solid lines correspond to the right-hand side of eq. (7.24). The triangles (squares) are the masses of the modes $\zeta(\theta)$ which are even (odd) under $\theta \to \pi - \theta$.

The equations of motion derived from (7.20), subjected to the boundary conditions (7.23) can be integrated numerically by means of the shooting technique. One first enforces the condition at $\theta = \theta_\Lambda$ and then varies the mass M until $\zeta(\pi - \theta_\Lambda)$ vanishes. This only happens for a discrete set of values of the mass M. For a given value of l, let us order the solutions in increasing value of the mass. In general one notices that the n^{th} mode has $n-1$ nodes in the interval $\theta_\Lambda < \theta < \pi - \theta_\Lambda$ and for n even (odd) the function $\zeta(\theta)$ is odd (even) under $\theta \to \pi - \theta$. In figure 5 we have plotted the first four modes corresponding to $r_* = 0.3$, $\Lambda = 3$ and $l = 0$. The modes odd under $\theta \to \pi - \theta$ vanish at $\theta = \pi/2$ and their masses and shapes match those found with the lagrangian (7.5) and the boundary condition (7.14). On the contrary, the modes with an even number of nodes in $\theta_\Lambda < \theta < \pi - \theta_\Lambda$ are the ones we were missing in the formulation in which r is taken as worldvolume coordinate.

Let $M_{n,l}(r_*, \Lambda)$ be the mass corresponding to the n^{th} mode for a given value l and the mass scales r_* and Λ. Our numerical results are compatible with an expression of $M_{n,l}(r_*, \Lambda)$ of the form:

$$M_{n,l}(r_*, \Lambda) = \sqrt{m^2(r_*, \Lambda) n^2 + l^2}\,. \tag{7.24}$$

To illustrate this fact we have plotted in figure 6 the values of $M_{n,l}(r_*, \Lambda)$ for $r_* = 0.3$ and $\Lambda = 3$, together with the curves corresponding to the right-hand side of (7.24).

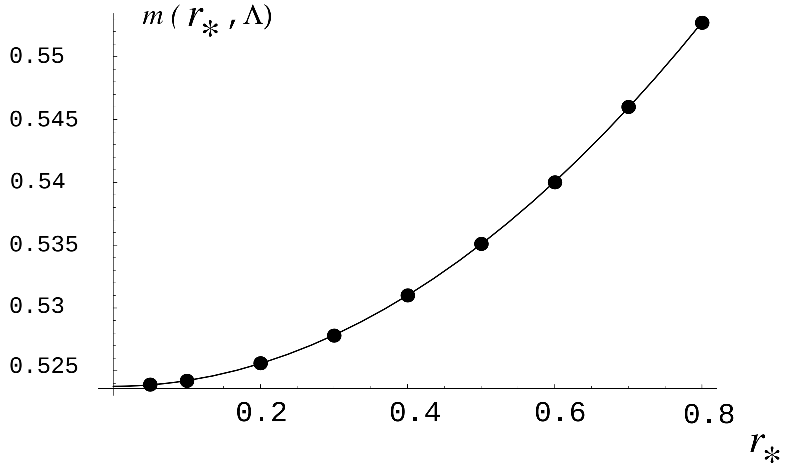

Figure 7: Dependence of $m(r_*, \Lambda)$ on r_* for $\Lambda = 3$. The solid line is a fit to the quadratic function (7.25).

We have also studied the dependence of the coefficient $m(r_*, \Lambda)$ on the two scales (r_*, Λ). Recall that r_* is the minimal separation of the brane probe from the origin and, thus, can be naturally identified with the mass of the quarks. We obtained that $m(r_*, \Lambda)$ can be represented by an expression of the type:

$$m(r_*, \Lambda) = \frac{\pi}{2\Lambda} + b(\Lambda)\, r_*^2 \,. \tag{7.25}$$

The coefficients on the right-hand side of eq. (7.25) have been obtained by a fit of $m(r_*, \Lambda)$ to a quadratic expression in r_*. An example of this fit is presented in figure 7. The first term in eq. (7.25) is a universal term (independent of r_*) which can be regarded as a finite size effect induced by our regularisation procedure. We also have determined the dependence of the coefficient b on Λ and it turns out that one can fit $b(\Lambda)$ to the expression $b(\Lambda) = 0.23\,\Lambda^{-2} + 0.53\,\Lambda^{-3}$

We have obtained a very regular mass spectrum of particles, classified by two quantum numbers n, l (see eqs. (7.24) and (7.25)). We can offer an interpretation of these formulas. Indeed, recall that the two-submanifold on which we are wrapping the brane is topologically like a cylinder, with the compact direction parametrized by the angle φ. The quantum number l is precisely the eigenvalue of the operator $-i\partial_\varphi$, which generates the shifts of φ and, indeed, the dependence on l of the mass displayed in eq. (7.24) is the typical one for a Kaluza-Klein reduction along a compact coordinate. Therefore, we should interpret the mesons with $l \neq 0$ as composed by Kaluza-Klein modes. Moreover, the term in (7.25) proportional to r_*^2 can be understood as the contribution coming from the mass of the constituent quarks, while the term $\pi/2\Lambda$ can be interpreted as a contribution coming from the 'finite size' of the meson. Indeed, it looks like a Casimir energy and it is originated in the presence of the cut-off Λ.

It is perhaps convenient to emphasize again that the cut-off procedure implemented here is a very natural procedure for this type of computations in this supergravity dual.

JHEP12(2003)024

In fact, given that the mesons are an IR effect in SQCD, we expect the contributions of high energy effects to be irrelevant or negligible in the physical properties of the meson itself. This is indeed what we are doing when cutting off the integration range. We are just taking into account the 'non-abelian' part of the probes, while neglecting the 'abelian' part or, equivalently, discarding high energy contributions.

7.2 Vector mesons

Let us now excite a gauge field that is living in the worldvolume of the brane. The linearized equations of motion are:

$$\partial_m \left[e^{-\phi} \sqrt{-g_{st}}\, \mathcal{F}^{nm} \right] = 0 \,, \tag{7.26}$$

where g_{st} is the determinant of the induced metric in the string frame and $\mathcal{F}^{nm}$ is the field strength of the worldvolume gauge field $\mathcal{A}_m$, i.e. $\mathcal{F}_{nm} = \partial_n \mathcal{A}_m - \partial_m \mathcal{A}_n$. Let us assume that the only non-vanishing components of the gauge field $\mathcal{A}$ are those along the unwrapped directions x^μ of the worldvolume of the brane. In what follows we are going to use θ as worldvolume coordinate. Let us put

$$\mathcal{A}_\mu(\theta, x, \varphi) = \epsilon_\mu \varsigma(\theta)\; e^{ikx}\; e^{il\varphi} \,, \tag{7.27}$$

where ϵ_μ is a constant polarization four-vector and l must be an integer. It follows from the equations of motion (7.26) that ϵ_μ must be transverse, i.e.:

$$k^\mu\, \epsilon_\mu = 0 \,, \tag{7.28}$$

and that $\varsigma(\theta)$ must satisfy the following second-order differential equation:

$$\partial_\theta \left[\frac{\sin\theta}{\tanh r_0}\, \partial_\theta\, \varsigma \right] + \frac{\tanh r_0}{\sin\theta} \left[M^2 \left(\cos^2\theta + r_0\, \coth r_0\, \sin^2\theta \right) - l^2 \right] \varsigma = 0 \,, \tag{7.29}$$

where, as in eq. (7.7), $k^2 = -M^2$. We have solved numerically this equation by means of the shooting technique with the boundary conditions

$$\varsigma(\theta_\Lambda) = \varsigma(\pi - \theta_\Lambda) = 0 \,. \tag{7.30}$$

Surprisingly, the set of possible values of M is given by the same expression as in the scalar meson case, i.e. eq. (7.24), with a coefficient function $m(r_*, \Lambda)$ which is, within the accuracy of our approximate calculation, equal numerically to that of the scalar mesons. This is quite remarkable since the differential equations we are solving in both cases are quite different (the equation satisfied by $\zeta(\theta)$ is much more complicated than eq. (7.29)). Thus, to summarize, we predict a degeneracy between the scalar and vector mesons in the corresponding $\mathcal{N} = 1$ SYM theory.

7.3 Gauge theory interpretation

Let us comment on some gauge theory aspects that can be read from the brane probes in sections 3 to 7. For this purpose, we shall concentrate on the main solutions that have been used, namely, the abelian solution with unit winding ($n = 1$) in eq. (5.22) and its non-abelian extension of eq. (6.15).

First of all, let us analyze the R-symmetry of the gauge theory from the probe viewpoint. It is clear that the abelian solution (5.22) is invariant under shifts of ψ by a constant since the value of this angle is an arbitrary constant in this solution. This symmetry has been identified as the geometric dual of the R-symmetry [6, 34, 35]. Actually, this is not a U(1) invariance of the background because [6, 34, 35] of the presence of the RR form that selects, by consistency, only some particular values of the angle ψ, i.e. $\psi = 2\pi n/N$ with $1 \leq n \leq 2N$. So, the abelian probes can see a $\mathbb{Z}_{2N}$ symmetry. In contrast, when we consider the non-abelian probe, the solution (6.15) selects two particular values of the angle ψ, thus breaking $\mathbb{Z}_{2N}$ down to $\mathbb{Z}_2$. Notice that the $\mathrm{U}(1) \to \mathbb{Z}_{2N}$ breaking is an UV effect (it takes place already in the abelian background) while the $\mathbb{Z}_{2N} \to \mathbb{Z}_2$ breaking is an IR effect which appears only when one considers the full non-abelian regular background. This same breaking pattern was observed in the case of SQCD with massive flavors. Indeed, as showed in [29], the theory with massive flavors has a non-anomalous discrete $\mathbb{Z}_{2N}$ R-symmetry, given by (component fields are used, λ is the gaugino and Φ and $\bar{\Phi}$ are the squarks):

$$\lambda \to e^{-i\pi n/N}\lambda\,, \qquad \Phi \to e^{-i\pi n/N}\Phi\,, \qquad \bar{\Phi} \to e^{-i\pi n/N}\bar{\Phi}\,, \tag{7.31}$$

with $n = 1, \ldots, 2N$

As shown in [29], this $\mathbb{Z}_{2N}$ symmetry is broken down to $\mathbb{Z}_2$ by the formation of a squark condensate. Indeed, one can see that $< \Phi\bar{\Phi} >$ transforms as $< \Phi\bar{\Phi} > \to e^{-2i\pi n/N} < \Phi\bar{\Phi} >$ leaving us with a $\mathbb{Z}_2$. Besides, the squark condensate is consistent with supersymmetry, because the F-term equation of motion $< \Phi\bar{\Phi} > -m = 0$ is satisfied. Notice that this preservation of symmetry when $m \neq 0$ is in agreement with the kappa symmetry of our brane probes.

Apart from all this, there is a vectorial U(1) symmetry that remains unbroken in our brane probe analysis and that we have associated with the invariance under translations in $\tilde{\varphi}$. On the field theory side this symmetry can be identified with a phase change of the full chiral/antichiral multiplet $\Phi \to e^{i\alpha}\Phi$, $\bar{\Phi} \to e^{-i\alpha}\bar{\Phi}$, which is nothing but the $\mathrm{U}(1)_B$ baryonic number symmetry of the theory. The two possible assignations ± 1 of the baryonic charge are in correspondence with the two possible identifications of the S^1 described by $\tilde{\varphi}$ with the S^1 parametrized by φ. The spectrum we have found is independent of this identification.

We would like also to comment briefly on the possibility of taking the parameter $r_* = 0$. Given that this parameter can be associated with the mass of the quark (since it is the characteristic distance between the probe brane and the background) one would like to study the case in which this parameter is taken to be zero. Nevertheless the approach implemented here seems to break down for this particular value of the parameter. Indeed, taking $r_* = 0$ will imply that $\theta_0 = 0$, so, a fluctuation of it can lead to negative values of θ taking us out of the range of this coordinate. The fact that our approach apparently does not work for the case of massless quarks seems to be in agreement with the fact that SQCD with massless flavors, has some special properties like spontaneous breaking of supersymmetry, the existence of a runaway potential (Affleck-Dine-Seiberg potential) and the non-existence of a well-defined vacuum state. Notice that, when $r_* \neq 0$, our approach, by construction, deals with massive quarks that preserve supersymmetry, since our probes were constructed by that requirement.

JHEP12(2003)024

8. Summary and conclusions

Let us summarize and repeat here some points that we believe are worth stressing again.

We studied a supergravity dual to $\mathcal{N} = 1$ SYM, based on a geometry representing a stack of $D5$ branes wrapping a two-cycle. In this gravity solution, we have found special surfaces where one can put probe D5-branes without spoiling supersymmetry. This was done, basically, by imposing that the Killing spinors of the background satisfy the projection imposed by the kappa symmetry of the D5-brane probe.

A wide variety of kappa symmetric solutions was found and a rich mathematical structure was pointed out. In fact, solutions where the probes are at a fixed distance from the "background branes" are shown to break supersymmetry. This phenomenon is in agreement with the non-existence of moduli space of $\mathcal{N} = 1$ SYM. On the other hand, we studied solutions corresponding to the abelian background (that is to say the large r regime of the full background). These abelian solutions are shown to have a very interesting analytic structure (harmonic functions and Cauchy-Riemann equations show up) allowing interesting explicit solutions. In extending these studies to the full background, we found several classes of non-abelian solitons. We think that it might be possible that these solutions show themselves useful when studying other aspects of the model not explored in this paper.

Then, in section 7, we have used the kappa symmetric solutions (in particular those that we called unit-winding) to introduce fundamental matter in the dual to SYM theory.

Given that the brane probes do not backreact on the background, these flavors are introduced in the so called quenched approximation. Nevertheless, many qualitative features and quantitative predictions for the strong coupling regime of $\mathcal{N} = 1$ SQCD can be addressed. Among them, the qualitative difference between the massless and massive flavors are clear in this picture. Indeed, by construction, our approach deals with the massive-flavor case, so, the problems or peculiarities of the massless case can be seen in this approach under a different geometrical perspective.

Other characteristic feature of SQCD with few flavors is the breaking pattern of R-symmetry. The $\mathbb{Z}_{2N} \to \mathbb{Z}_2$ breaking is geometrically very clear from the brane probe perspective. Also, the preserved $\mathrm{U}(1)_B$ is geometrically realized, as explained in the previous section, by arbitrary changes in the coordinate $\tilde{\varphi}$.

A mass spectrum for the low energy excitations (mesons) of massive SQCD was found and we believe this is a very interesting and quantitative prediction of this paper. In fact, a nice formula for the masses is derived that exhibits a BPS-like behavior with the level (n) and with the Kaluza-Klein quantum number (l) of the meson. Basically, our formula gives the meson masses in terms of the mass of the fundamentals and the Casimir energy due to finite size of the meson, and shows explicitly the contamination of the meson spectrum due to the Kaluza-Klein modes. It would be very interesting if lattice calculations could validate or invalidate the formula found here.

The appendices provide a hopefully very useful technical information for other workers in the area which contain, in particular, the explicit expressions of the Killing spinors of the background considered here and for another related background dual to $\mathcal{N} = 1$ SYM theory with a Chern-Simons term in $2 + 1$ dimensions.

We believe that this paper opens some interesting routes of future research. To mention some of them, it should be interesting to repeat the procedure here in other "wrapped branes" setups, dual to lower supersymmetric field theories, to see if the features for the mass formula and symmetries are preserved. In particular, it would be very interesting to apply the methods employed here to the Klebanov-Strassler background [23]. The analysis of ref. [36] of the glueball spectrum suggests that, contrary to what happens to the background studied here, the spectrum of mesons could be obtained without introducing any additional cut-off. Actually, this meson spectrum was obtained in ref. [22] from the fluctuations of a D7-brane probe wrapped on a submanifold which is analogous to our cylinders. Presumably, our method would allow us to obtain a family of supersymmetric embeddings which depend on a parameter that can be identified with the distance between the brane probe and the origin.

On the other hand, the brane probe lagrangian should be regarded as a sort of "low energy effective lagrangian" for the strong coupling limit of SQCD with few flavors; it would be very nice to develop the physics of this effective theory because, possibly, the scattering of mesons and other interesting features, like low energy theorems, could be studied or understood in this geometrical context.

Extending the treatment of this paper to non supersymmetric theories might seem a contradiction (the paper is based on kappa symmetry) but it is likely that with softly broken backgrounds like those appeared recently in the literature, see ref. [12], similar methods to the ones exhibited here can be used.

Acknowledgments

We are grateful to D. Crooks, J. D. Edelstein, J. Erdmenger, S. A. Hartnoll, R. Hernández, M. Kruczenski, J. M. Maldacena, D. & T. Mateos, J. Mas, J. W. Negele, L. Pando-Zayas, R. Portugues, K. Rajagopal, M. Schvellinger and D. Tong for useful discussions and encouragement. We have benefited of the nice atmosphere of the Benasque Center for Science, where part of this work has been done. C. N. is supported by a Pappalardo Fellowship, and in part by funds provided by the U.S. Department of Energy (D.O.E) under cooperative research agreement # DF-FC02-94ER40818. A. P. and A. V. R are supported in part by MCyT, FEDER and Xunta de Galicia under grant BFM2002-03881 and by the EC Commission under the FP5 grant HPRN-CT-2002-00325.

A. Supersymmetry of the $\mathcal{N}=1$ supergravity duals

The supersymmetry transformations for the type IIB dilatino λ and gravitino Ψ_μ when the RR three-form is nonzero are:

$$\begin{aligned}\delta\lambda &= \frac{i}{2}\partial_\mu\,\phi\,\Gamma^\mu\,\epsilon^* + \frac{1}{24}\,e^{\frac{\phi}{2}}\,F^{(3)}_{\mu_1\mu_2\mu_3}\,\Gamma^{\mu_1\mu_2\mu_3}\,\epsilon\,,\\ \delta\Psi_\mu &= D_\mu\epsilon + \frac{i}{96}\,e^{\frac{\phi}{2}}\,F^{(3)}_{\mu_1\mu_2\mu_3}\,(\Gamma_\mu^{\;\mu_1\mu_2\mu_3} - 9\,\delta_\mu^{\mu_1}\,\Gamma^{\mu_2\mu_3})\,\epsilon^*\,.\end{aligned} \tag{A.1}$$

The Killing spinors of the background are those ϵ for which the right-hand side of eq. (A.1) vanishes. In order to satisfy the equations $\delta\lambda = \delta\Psi_\mu = 0$ we will have to impose certain projection conditions to ϵ. Once these projections are imposed, the equations $\delta\lambda = \delta\Psi_\mu = 0$ are equivalent to a system of first-order differential equations for the metric and RR three-form. Moreover, following the methodology of ref. [16] (see also ref. [8]), we will be able to determine the explicit form of ϵ from the projections that satisfies. In this appendix we will carry out this program for two $\mathcal{N} = 1$ supergravity duals which correspond to D5-branes wrapping a two- and a three-cycle.

A.1 Fivebranes wrapped on a two-cycle

For a metric ansatz such as that of eq. (2.1), let us consider the frame

$$\begin{aligned} e^{x^i} &= e^{\frac{\phi}{4}}\, dx^i\,, & &(i = 0,1,2,3)\,, \\ e^1 &= e^{\frac{\phi}{4}+h}\, d\theta\,, & e^2 &= e^{\frac{\phi}{4}+h}\, \sin\theta d\varphi\,, \\ e^r &= e^{\frac{\phi}{4}}\, dr\,, & e^{\hat{i}} &= \frac{e^{\frac{\phi}{4}}}{2}(w^i - A^i)\,, \qquad (i = 1,2,3)\,. \end{aligned} \tag{A.2}$$

In order to solve the equations $\delta\lambda = \delta\Psi_\mu = 0$, we impose to the spinor ϵ the condition for a SUSY two-cycle:

$$\Gamma_{12}\,\epsilon = \hat{\Gamma}_{12}\,\epsilon\,, \tag{A.3}$$

and the following projection

$$\epsilon = i\epsilon^*\,. \tag{A.4}$$

Then, the condition $\delta\lambda = 0$ yields the equation

$$\phi'\,\epsilon - \left(1 + \frac{e^{-2h}}{4}\,(a^2 - 1)\right)\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon + \frac{1}{2}\,a'\,e^{-h}\,\Gamma_2\hat{\Gamma}_2\epsilon = 0\,. \tag{A.5}$$

Let us now consider the supersymmetric variation of the different components of the gravitino. It is easy to verify that, with the projections (A.3) and (A.4), the equation corresponding to the unwrapped directions x^i and that corresponding to the directions $\hat{\imath}$, coincide with the one obtained from the variation of the dilatino (eq. (A.5)). On the other hand the variation of the components along the two-sphere gives rise to the following equation

$$h'\epsilon - \frac{1}{2}\,a'\,e^{-h}\,\Gamma_2\hat{\Gamma}_2\epsilon - a\,e^{-h}\,\Gamma_2\hat{\Gamma}_2\Gamma_r\hat{\Gamma}_{123}\,\epsilon + \frac{1}{2}\,(a^2 - 1)\,e^{-2h}\,\Gamma_r\hat{\Gamma}_{123}\,\epsilon = 0\,, \tag{A.6}$$

whereas, after using eq. (A.5), the condition $\delta\psi_r = 0$ is equivalent to

$$\partial_r\epsilon - \frac{1}{2}\,a'\,e^{-h}\,\Gamma_2\hat{\Gamma}_2\,\epsilon - \frac{1}{8}\phi'\,\epsilon = 0\,. \tag{A.7}$$

Moreover, from the variation of the dilatino (eq. (A.5)) we learn that ϵ must satisfy

$$\Gamma_r\hat{\Gamma}_{123}\,\epsilon = (\beta + \tilde{\beta}\Gamma_2\hat{\Gamma}_2)\,\epsilon\,, \tag{A.8}$$

where β and $\tilde{\beta}$ can be read from eq. (A.5), namely

$$\beta = \frac{\phi'}{1 + \frac{e^{-2h}}{4}\,(a^2 - 1)}\,, \qquad \tilde{\beta} = \frac{1}{2}\,\frac{e^{-h}a'}{1 + \frac{e^{-2h}}{4}\,(a^2 - 1)}\,. \tag{A.9}$$

On the other hand, from the condition $(\Gamma_r\hat{\Gamma}_{123})^2\epsilon = \epsilon$ and since $\{\Gamma_r\hat{\Gamma}_{123}, \Gamma_2\hat{\Gamma}_2\} = 0$, it follows that:

$$\beta^2 + \tilde{\beta}^2 = 1\,, \tag{A.10}$$

and, therefore, we can represent β and $\tilde{\beta}$ as:

$$\beta = \cos\alpha\,, \qquad \tilde{\beta} = \sin\alpha\,. \tag{A.11}$$

Substituting the radial projection (eq. (A.8)) into eq. (A.6) and considering the terms with and without $\Gamma_2\hat{\Gamma}_2$, we get the following two equations:

$$h' = -\frac{1}{2}\,e^{-2h}\,(a^2-1)\,\beta - a e^{-h}\tilde{\beta}\,, \qquad a' = -2a\beta + e^{-h}\,(a^2-1)\,\tilde{\beta}\,. \tag{A.12}$$

By using the definition of β and $\tilde{\beta}$ (eq. (A.9)) into the second equation in (A.12), we get the following relation between ϕ' and a':

$$\phi' = -\frac{a'}{2a}\left[1 - \frac{1}{4}\,e^{-2h}\,(a^2-1)\right]. \tag{A.13}$$

Furthermore, from the condition $\beta^2 + \tilde{\beta}^2 = 1$ one obtains a new relation between ϕ' and a', namely:

$$\phi'^2 + \frac{1}{4}\,e^{-2h}\,a'^2 = \left[1 + \frac{1}{4}\,e^{-2h}\,(a^2-1)\right]^2. \tag{A.14}$$

By combining eqs. (A.13) and (A.14) one can get the expression of ϕ' and a' in terms of a and h. Moreover, by using these results in eq. (A.9), one can get β and $\tilde{\beta}$ as functions of a and h and, by plugging the corresponding expressions on the first eq. in (A.12), one can obtain the differential equation for h. In order to write these expressions in a compact form, let us define:

$$Q \equiv \sqrt{e^{4h} + \frac{1}{2}\,e^{2h}\,(a^2+1) + \frac{1}{16}\,(a^2-1)^2}\,. \tag{A.15}$$

Then, one has the following system of first-order differential equations for ϕ, h and a:

$$\begin{aligned}
\phi' &= \frac{1}{Q}\left[e^{2h} - \frac{e^{-2h}}{16}\,(a^2-1)^2\right],\\
h' &= \frac{1}{2Q}\left[a^2 + 1 + \frac{e^{-2h}}{4}\,(a^2-1)^2\right],\\
a' &= -\frac{2a}{Q}\left[e^{2h} + \frac{1}{4}\,(a^2-1)\right],
\end{aligned} \tag{A.16}$$

and the values of $\beta = \cos\alpha$ and $\tilde{\beta} = \sin\alpha$, which are given by:

$$\sin\alpha = -\frac{a e^h}{Q}\,, \qquad \cos\alpha = \frac{e^{2h} - \frac{1}{4}\,(a^2-1)}{Q}\,. \tag{A.17}$$

It is interesting to notice that, when solving the quadratic eq. (A.14) to obtain (A.16) and (A.17), we have a sign ambiguity. We have fixed this sign by requiring that h' is always positive.

It remains to verify the fulfillment of equation (A.7). Notice first of all that the radial projection (A.8) can be written as

$$\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon = e^{\alpha\Gamma_2\hat{\Gamma}_2}\,\epsilon\,, \tag{A.18}$$

which, after taking into account that $\{\Gamma_r\hat{\Gamma}_{123}, \Gamma_2\hat{\Gamma}_2\} = 0$, can be solved as:

$$\epsilon = e^{-\frac{\alpha}{2}\Gamma_2\hat{\Gamma}_2}\epsilon_0\,, \qquad \Gamma_r\,\hat{\Gamma}_{123}\,\epsilon_0 = \epsilon_0\,. \tag{A.19}$$

Inserting this parametrization of ϵ into eq. (A.7), we get two types of terms, with and without $\Gamma_2\hat{\Gamma}_2$, which yield the following two equations:

$$\partial_r\epsilon_0 - \frac{1}{8}\,\phi'\,\epsilon_0 = 0\,, \qquad \alpha' = -e^{-h}\,a'\,. \tag{A.20}$$

The first of these equations can be solved immediately, namely

$$\epsilon_0 = e^{\frac{\phi}{8}}\,\eta\,, \tag{A.21}$$

with η being a constant spinor. Moreover, by differentiating eq. (A.17) and using eq. (A.16), one can verify that the second equation in (A.20) is satisfied. Thus, the explicit form of the Killing spinor is

$$\epsilon = e^{\frac{\alpha}{2}\Gamma_1\hat{\Gamma}_1}\;e^{\frac{\phi}{8}}\,\eta\,, \tag{A.22}$$

where η is a constant spinor satisfying:

$$\Gamma_{x^0\cdots x^3}\,\Gamma_{12}\,\eta = \eta\,, \qquad \Gamma_{12}\,\eta = \hat{\Gamma}_{12}\,\eta\,, \qquad \eta = i\eta^*\,. \tag{A.23}$$

eq. (A.16) is a system of first-order differential equations whose solution determines the metric, dilaton and RR three-form of the background. It can be verified that the values given in eq. (2.4) solve the system (A.16). Moreover, by plugging in eq. (A.15) the values of h and a given in eq. (2.4), one verifies that

$$Q = r\,. \tag{A.24}$$

Then, one gets the following simple expression for $\cos\alpha$:

$$\cos\alpha = \coth 2r - \frac{2r}{\sinh^2 2r}\,. \tag{A.25}$$

On the other hand, being a spinor of definite chirality of type IIB supergravity, ϵ satisfies $\Gamma_{x^0\cdots x^3}\Gamma_{12}\Gamma_r\hat{\Gamma}_{123}\epsilon = \epsilon$. Thus, if we multiply the radial projection condition (A.8) by $\Gamma_{x^0\cdots x^3}\Gamma_{12}$, we obtain:

$$\Gamma_{x^0\cdots x^3}(\cos\alpha\,\Gamma_{12} + \sin\alpha\,\Gamma_1\hat{\Gamma}_2)\,\epsilon = \epsilon\,. \tag{A.26}$$

eqs. (A.3), (A.4) and (A.26) are the ones needed in the analysis of the kappa symmetry of a D5-brane probe (eq. (2.12)).

JHEP12(2003)024

A.2 Fivebranes wrapped on a three-cycle

As a second example of the calculation of the Killing spinors of a $\mathcal{N}=1$ supergravity dual, let us consider the supergravity solution for a D5 brane wrapped on a three-sphere. In this case the ansatz for the metric in Einstein frame is

$$ds^2 = e^{\frac{\phi}{2}} \left[dx_{1,2}^2 + \frac{1}{4} R(r)^2 (\sigma^i)^2 + dr^2 + \frac{1}{4} (w^i - A^i)^2 \right], \tag{A.27}$$

where ϕ is the dilaton, $R(r)$ is a function to be determined and σ^i and w^i $(i = 1,2,3)$ are two sets of $su(2)$ left invariant one-forms. The gauge field components A^i are parametrized in terms of a new function $\omega(r)$ as:

$$A^i = \frac{1+\omega(r)}{2}\,\sigma^i\,. \tag{A.28}$$

Let F^i be the field strength of A^i (defined as in eq. (2.6)). Its components are:

$$F^i = \frac{\omega'}{2}\,dr\wedge\sigma^i + \frac{\omega^2-1}{8}\,\epsilon_{ijk}\,\sigma^j\wedge\sigma^k\,. \tag{A.29}$$

The RR three-form of the background is now

$$F_{(3)} = -\frac{1}{4}(w^1-A^1)\wedge(w^2-A^2)\wedge(w^3-A^3) + \frac{1}{4}\sum_i F^i\wedge(w^i-A^i) + h\,, \tag{A.30}$$

where h is determined by requiring that $F_{(3)}$ satisfies the Bianchi identity $dF_{(3)} = 0$. One easily verifies that h must satisfy the following equation

$$dh = \frac{1}{4}\sum_i F^i\wedge F^i\,. \tag{A.31}$$

By explicit calculation, one gets:

$$\sum_i F^i\wedge F^i = \frac{1}{8}(\omega^2-1)\,\omega'\,\epsilon_{jkm}\,dr\wedge\sigma^j\wedge\sigma^k\wedge\sigma^m\,. \tag{A.32}$$

Thus, the equation for h can be solved as:

$$h = \frac{1}{32}\,\frac{1}{3!}\,V(r)\,\epsilon_{ijk}\,\sigma^i\wedge\sigma^j\wedge\sigma^k\,, \tag{A.33}$$

where

$$V(r) = 2\omega^3 - 6\omega + 8k\,, \tag{A.34}$$

with k being a constant.

Let us study the supersymmetry preserved by this ansatz in the frame:

$$\begin{aligned} e^{x^i} &= e^{\frac{\phi}{4}}\,dx^i\,, && (i=0,1,2)\,, && e^r = e^{\frac{\phi}{4}}\,dr\,,\\ e^i &= \frac{1}{2}e^{\frac{\phi}{4}}\,R\,\sigma^i\,, && e^{\hat{i}} = \frac{1}{2}e^{\frac{\phi}{4}}\,(w^i-A^i)\,, && (i=1,2,3)\,. \end{aligned} \tag{A.35}$$

JHEP12(2003)024

We now impose the following projections on the spinors

$$\begin{aligned}\Gamma_1\hat{\Gamma}_1\,\epsilon &= \Gamma_2\hat{\Gamma}_2\,\epsilon = \Gamma_3\hat{\Gamma}_3\,\epsilon\,,\\ \epsilon &= i\epsilon^*\,.\end{aligned} \tag{A.36}$$

Then, the condition $\delta\lambda = 0$ becomes

$$\phi'\,\epsilon - \left(1 + 3\,\frac{\omega^2-1}{4R^2}\right)\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon + \frac{3\omega'}{4R}\,\Gamma_1\hat{\Gamma}_1\epsilon - \frac{V}{8R^3}\,\Gamma_1\hat{\Gamma}_1\,\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon = 0\,. \tag{A.37}$$

Moreover, from $\delta\psi_i = 0$, one gets

$$R'\epsilon - \frac{\omega'}{2}\,\Gamma_1\hat{\Gamma}_1\epsilon + \frac{\omega^2-1}{R}\,\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon + \left(\frac{V}{8R^2} - \omega\right)\Gamma_1\hat{\Gamma}_1\,\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon = 0\,. \tag{A.38}$$

The vanishing of the supersymmetric variation of the radial component of the gravitino gives rise to:

$$\partial_r\epsilon - \frac{3\omega'}{4R}\,\Gamma_1\hat{\Gamma}_1\epsilon - \frac{1}{8}\,\phi'\,\epsilon = 0\,, \tag{A.39}$$

where we have used eq. (A.37). Let us solve these equations by taking the projection

$$\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon = (\beta + \tilde{\beta}\Gamma_1\hat{\Gamma}_1)\,\epsilon\,. \tag{A.40}$$

As in case studied in appendix A.1, from $(\Gamma_r\,\hat{\Gamma}_{123})^2\,\epsilon = \epsilon$ and $\{\Gamma_r\,\hat{\Gamma}_{123}, \Gamma_1\hat{\Gamma}_1\} = 0$ it follows that $\beta^2 + \tilde{\beta}^2 = 1$ and thus we can take $\beta = \cos\alpha$ and $\tilde{\beta} = \sin\alpha$.

Let us substitute our ansatz for $\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon$ (eq. (A.40)) on the equations coming from the dilatino and gravitino (eqs. (A.37) and (A.38)). From the terms containing the unit matrix, we obtain equations for ϕ' and R':

$$\begin{aligned}\phi' &= \left(1 + 3\,\frac{\omega^2-1}{4R^2}\right)\beta - \frac{V}{8R^3}\,\tilde{\beta}\,,\\ R' &= \frac{1-\omega^2}{R}\,\beta + \left(\frac{V}{8R^2} - \omega\right)\tilde{\beta}\,.\end{aligned} \tag{A.41}$$

Moreover, by considering the terms with $\Gamma_1\hat{\Gamma}_1$, we obtain two expressions for ω'

$$\begin{aligned}3\omega' &= \frac{V}{2R^2}\,\beta + \left(4R + 3\,\frac{\omega^2-1}{R}\right)\tilde{\beta}\,,\\ \omega' &= \left(\frac{V}{4R^2} - 2\omega\right)\beta + 2\,\frac{\omega^2-1}{R}\,\tilde{\beta}\,.\end{aligned} \tag{A.42}$$

By combining these last two equations we get

$$\left(\frac{V}{24R^2} - w\right)\beta = \left(\frac{1-w^2}{2R} + \frac{2R}{3}\right)\tilde{\beta}\,. \tag{A.43}$$

By plugging this last relation in the condition $\beta^2 + \tilde{\beta}^2 = 1$ one can easily obtain the expression of β and $\tilde{\beta}$. Indeed, let us define M as follows:

$$M \equiv \left(\frac{V}{24R^2} - w\right)^2 + \left(\frac{1-w^2}{2R} + \frac{2R}{3}\right)^2. \tag{A.44}$$

In terms of this new quantity M, the coefficients β and $\tilde{\beta}$ are given by:

$$\beta = \cos\alpha = \frac{1}{\sqrt{M}}\left(\frac{2R}{3} + \frac{1-\omega^2}{2R}\right), \qquad \tilde{\beta} = \sin\alpha = \frac{1}{\sqrt{M}}\left(\frac{V}{24R^2} - \omega\right). \tag{A.45}$$

By using these values of β and $\tilde{\beta}$ in the equations which determine R', ω' and ϕ', we obtain a system of first-order BPS equations which are identical to those written in refs. [31, 32, 33] (see also ref. [37]). They are:

$$\begin{aligned}
R' &= \frac{1}{3\sqrt{M}}\left[\frac{V^2}{64R^4} + \frac{1}{2R^2}(3(1-\omega^2)^2 - V\omega) + \omega^2 + 2\right], \\
\omega' &= \frac{4R}{3\sqrt{M}}\left[\frac{V}{32R^4}\,(1-\omega^2) + \frac{2k-\omega^3}{2R^2} - \omega\right], \\
\phi' &= -\frac{3}{2}(\log R)' + \frac{3}{2}\,\frac{\sqrt{M}}{R}\,.
\end{aligned} \tag{A.46}$$

The radial projection condition (A.40) can be written as

$$\Gamma_r\,\hat{\Gamma}_{123}\,\epsilon = e^{\alpha\Gamma_1\hat{\Gamma}_1}\,\epsilon\,. \tag{A.47}$$

Since $\{\Gamma_r\,\hat{\Gamma}_{123}, \Gamma_1\hat{\Gamma}_1\} = 0$, this equation can be solved as:

$$\epsilon = e^{-\frac{\alpha}{2}\Gamma_1\hat{\Gamma}_1}\,\epsilon_0\,, \qquad \Gamma_r\,\hat{\Gamma}_{123}\,\epsilon_0 = \epsilon_0\,. \tag{A.48}$$

Plugging now this parametrization of ϵ into the equation obtained from the variation of the radial component of the gravitino (eq. (A.39)), we arrive at the following two equations:

$$\partial_r\,\epsilon_0 = \frac{\phi'}{8}\,\epsilon_0\,, \qquad \alpha' = -\frac{3}{2}\,\omega'\,R\,. \tag{A.49}$$

The equation for ϵ_0 can be solved immediately:

$$\epsilon_0 = e^{\frac{\phi}{8}}\,\eta\,, \tag{A.50}$$

where η is a constant spinor. Moreover, one can verify that the equation for α is a consequence of the first-order BPS equations (A.46). Therefore, the Killing spinors for this geometry are of the form:

$$\epsilon = e^{-\frac{\alpha}{2}\Gamma_1\hat{\Gamma}_1}e^{\frac{\phi}{8}}\,\eta\,, \tag{A.51}$$

where η is constant and satisfies the following conditions:

$$\begin{aligned}
\Gamma_{x^0\cdots x^2}\,\Gamma_{123}\,\eta &= \eta\,, \\
\Gamma_1\hat{\Gamma}_1\,\eta &= \Gamma_2\hat{\Gamma}_2\,\eta = \Gamma_3\hat{\Gamma}_3\,\eta\,, \\
\eta &= i\eta^*\,.
\end{aligned} \tag{A.52}$$

Notice that, for the spinor ϵ, the first of these projections can be recast in the form:

$$\Gamma_{x^0\cdots x^2}(\cos\,\alpha\Gamma_{123} - \sin\alpha\hat{\Gamma}_{123})\,\epsilon = \epsilon\,. \tag{A.53}$$

B. Asymptotic behaviour of the fluctuations

This appendix is devoted to the determination of the asymptotic form of the fluctuations around the kappa-symmetric static configurations of the D5-brane probe. In appendix B.1 we will consider the large and small r behaviour of the solutions of the equation of motion corresponding to the lagrangian (7.5), which describes the small oscillations around the unit-winding embedding (7.2). In appendix B.2 we will study the asymptotic form of the fluctuations of the n-winding embedding in the abelian background. Following our general arguments, the UV behaviour of the abelian and non-abelian fluctuations must be the same and, thus, from this analysis we can get an idea of the nature of the small oscillations around the general non-abelian n-winding configurations (whose analytical form we have not determined) for large values of the radial coordinate.

B.1 Non abelian unit-winding embedding

For large r the lagrangian (7.5) takes the form:

$$\mathcal{L} = -\frac{1}{2}\, e^{2\phi}\, [\, r\, (\partial_r \chi)^2 + 2r\, \chi\, \partial_r \chi + r\chi^2 + r\, (\partial_x \chi)^2 + r\, (\partial_\varphi \chi)^2\,]\,, \tag{B.1}$$

where we have not expanded the dilaton and we have eliminated the exponentially suppressed terms. Using the asymptotic value of $\partial_r \phi$:

$$\partial_r \phi \,\approx\, 1 \,-\, \frac{1}{4r-1}\,, \tag{B.2}$$

we obtain the following equation for ξ:

$$\partial_r^2\, \xi + \frac{8r^2-1}{4r^2-r}\, \partial_r\, \xi + \left(M^2 - l^2 + 1 + \frac{2r-1}{4r^2-r} \right) \xi = 0\,. \tag{B.3}$$

To study the UV behaviour of the solutions of this differential equation it is interesting to rewrite it with the different coefficient functions expanded in powers of $1/r$ as:

$$\partial_r^2\, \xi + \left(a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots \right) \partial_r\, \xi + \left(b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots \right) \xi = 0\,, \tag{B.4}$$

where

$$a_0 = 2\,, \qquad b_0 = M^2 - l^2 + 1\,, \qquad a_1 = b_1 = \frac{1}{2}\,, \qquad a_2 = b_2 = -\frac{1}{8}\,, \cdots \tag{B.5}$$

We want to solve eq. (B.4) by means of an asymptotic Frobenius expansion of the type $\xi = r^\rho\, (c_0 + c_1/r + \cdots)$ for some exponent ρ. By substituting this expansion on the right-hand side of eq. (B.4) and comparing the terms with the different powers of r, we notice that there is only one term with r^ρ, namely $b_0 c_0 r^\rho$, which cannot be canceled. In order to get rid of this term, let us define a new function w as

$$\xi = e^{\alpha r}\, w\,, \tag{B.6}$$

with α being a number to be determined. The equation satisfied by w is the same as that of ξ with the changes:

$$\begin{aligned} a_0 &\to a_0 + 2\alpha\,, & b_0 &\to \alpha^2 + \alpha a_0 + b_0\,, \\ a_i &\to a_i\,, & b_i &\to b_i + \alpha a_i\,, \qquad (i = 1, 2, \cdots)\,. \end{aligned} \tag{B.7}$$

It is clear that we must impose the condition:

$$\alpha^2 + \alpha a_0 + b_0 = 0\,, \tag{B.8}$$

which determines the values of α. Writing now $w = r^{\rho}\,(c_0 + c_1/r + \cdots)$ and looking at the highest power of r in the equation of w (i.e. $\rho - 1$), we immediately obtain the value of ρ, namely:

$$\rho = -\frac{\alpha a_1 + b_1}{2\alpha + a_0}\,, \tag{B.9}$$

and, clearly, as $r \to \infty$ the asymptotic behaviour of $\xi(r)$ is:

$$\xi(r) \approx e^{\alpha r}\, r^{\rho}\left(1 + o\left(\frac{1}{r}\right)\right). \tag{B.10}$$

In our case, it is easy to verify that the values of α and ρ are:

$$\alpha = -1 \pm i\sqrt{M^2 - l^2}\,, \qquad \rho = -\frac{1}{4}\,. \tag{B.11}$$

Then, it is clear that we have two independent behaviours for the real function $\xi(r)$, namely:

$$\xi(r) \sim \frac{e^{-r}}{r^{\frac{1}{4}}}\cos(\sqrt{M^2 - l^2}r)\,, \qquad \frac{e^{-r}}{r^{\frac{1}{4}}}\sin(\sqrt{M^2 - l^2}r)\,. \tag{B.12}$$

Notice that all solutions decrease exponentially when $r \to \infty$.

Let us now turn to the analysis of the fluctuations for small values of the radial coordinate. Recall that $r_* \le r \le \infty$. Near r_*, one can expand $\sin\theta_0$ and $\cos\theta_0$ as follows:

$$\begin{aligned} \sin\theta_0 &\approx 1 - \coth r_*\,(r - r_*) + \frac{1}{2}\,\frac{1 + \cosh^2 r_*}{\sinh^2 r_*}\,(r - r_*)^2 + \cdots\,, \\ \cos\theta_0 &\approx \sqrt{2\coth r_* (r - r_*)}\left[1 - \frac{\coth r_*}{2}\left(1 + \frac{1}{2\cosh^2 r_*}\right)(r - r_*) + \cdots\right]. \end{aligned} \tag{B.13}$$

Using these expressions it is straightforward to show that the lagrangian density of the quadratic fluctuations is given by:

$$\mathcal{L} = -\frac{1}{2}\, e^{2\phi_*}\,[A(r)\,(\partial_r\chi)^2 + B(r) 2\chi\partial_r\chi + C(r)\,\chi^2 + D(r)\,(\partial_x\chi)^2 + E(r)\,(\partial_\varphi\chi)^2]\,, \tag{B.14}$$

where $\phi_* = \phi(r_*)$ and the functions A, B, C, D and E are of the form:

$$\begin{aligned} A(r) &= [\,8\tanh r_*\,(r - r_*)^3\,]^{\frac{1}{2}}(1 + \mathcal{A}(r))\,, \\ B(r) &= \frac{\sqrt{2(r - r_*)}}{\sqrt{\coth r_*}}(1 + \mathcal{B}(r))\,, \\ C(r) &= \frac{1}{\sqrt{2\coth r_*\,(r - r_*)}}(1 + \mathcal{C}(r))\,, \\ D(r) &= \frac{r_*}{\sqrt{\coth r_*}}\,\sqrt{2(r - r_*)}(1 + \mathcal{D}(r))\,, \\ E(r) &= [\,\tanh r_*\,]^{\frac{3}{2}}\sqrt{2(r - r_*)}\,(1 + \mathcal{E}(r))\,. \end{aligned} \tag{B.15}$$

The functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$, satisfy:

$$\mathcal{A}(r), \mathcal{B}(r), \mathcal{C}(r), \mathcal{D}(r)\, \mathcal{E}(r) \sim o(r - r_*)\,. \tag{B.16}$$

Notice that, remarkably, after integrating by parts the $\chi \partial_r \chi$ term in the lagrangian, the singular term of $C(r)$ cancels against the leading term of $B(r)$. The equation of motion for ξ near r_* is:

$$\partial_r^2 \xi + \frac{A'(r)}{A(r)} \partial_r \xi + \frac{B'(r) - C(r) + M^2 D(r) - l^2 E(r)}{A(r)} \xi = 0\,. \tag{B.17}$$

In order to solve this equation in a power series expansion in $r - r_*$, it is important to understand the singularities of the different coefficients near r_*. It is immediate that:

$$\frac{A'(r)}{A(r)} = \frac{3}{2} \frac{1}{r - r_*} + \frac{\mathcal{A}'(r)}{1 + \mathcal{A}(r)} = \frac{3}{2} \frac{1}{r - r_*} + \text{regular}\,. \tag{B.18}$$

Similarly, the coefficient of ξ has a simple pole near r_*:

$$\begin{aligned} \frac{B'(r) - C(r) + M^2 D(r) - l^2 E(r)}{A(r)} &= \frac{3\mathcal{B}'(r_*) - \mathcal{C}'(r_*) + 2M^2 r_* - 2l^2 \tanh r_*}{4} \times \\ &\quad \times \frac{1}{r - r_*} + \text{regular}\,. \end{aligned} \tag{B.19}$$

It follows that the point $r = r_*$ is a singular regular point. The corresponding Frobenius expansion reads:

$$\xi(r) = (r - r_*)^\lambda \sum_{n=0}^{\infty} c_n\, (r - r_*)^n\,, \tag{B.20}$$

where λ satisfies the indicial equation, which can be obtained by plugging the expansion in the equation and looking at the term with lowest power of $r - r_*$ (i.e. $\lambda - 2$). In our case, λ must be a root of the quadratic equation:

$$\lambda(\lambda - 1) + \frac{3}{2}\lambda = 0\,, \tag{B.21}$$

i.e.:

$$\lambda = 0, -\frac{1}{2}\,. \tag{B.22}$$

This means that there are two independent solutions of the differential equation which can be represented by a Frobenius series around $r = r_*$, one of then is regular as $r \to r_*$ (the one corresponding to $\lambda = 0$), whereas the other diverges as $(r - r_*)^{-\frac{1}{2}}$.

B.2 n-Winding embeddings in the abelian background

Let us consider the abelian background and an embedding with winding number n, for which $\partial_\varphi \tilde{\varphi} = n$, $\sin\theta \partial_\theta \tilde{\theta} = n \sin\tilde{\theta}$ and $\psi = \psi_0 = \text{constant}$. Let us define

$$\begin{aligned} V &\equiv \frac{n}{2} \frac{\cos\tilde{\theta}}{\sin\theta} + \frac{1}{2}\cot\theta = \frac{n}{2\sin\theta} \frac{(1+\cos\theta)^n - (1-\cos\theta)^n}{(1+\cos\theta)^n + (1-\cos\theta)^n} + \frac{1}{2}\cot\theta\,, \\ W &\equiv \frac{n}{2} \frac{\sin\tilde{\theta}}{\sin\theta} = \frac{n(\sin\theta)^{(n-1)}}{(1+\cos\theta)^n + (1-\cos\theta)^n}\,. \end{aligned} \tag{B.23}$$

Choosing r as worldvolume coordinate and considering embeddings in which θ depends on both r and on the unwrapped coordinates x, we obtain the following lagrangian:

$$\mathcal{L} = -e^{2\phi}\,\sin\theta\Bigg[\sqrt{(e^{2h}+V^2+W^2)(1+(e^{2h}+W^2)((\partial_r\theta)^2+(\partial_x\theta)^2))}\;+ \\ +\,(e^{2h}+W^2)\partial_r\theta - V\Bigg]\,. \tag{B.24}$$

We shall expand this lagrangian around a configuration $\theta_0(r)$ such that

$$e^{2(r-r_*)} = \frac{1}{2}\,\frac{(1+\cos\theta_0(r))^n+(1-\cos\theta_0(r))^n}{(\sin\theta_0(r))^{n+1}}\,. \tag{B.25}$$

Notice that $\theta_0(r)$ remains unchanged under the transformation $n\to -n$. Thus, without loss of generality we shall restrict ourselves from now on to the case $n\geq 0$. Let us now define

$$V_0(r)\equiv V|_{\theta(r)=\theta_0(r)}\,,\qquad W_0(r)\equiv W|_{\theta(r)=\theta_0(r)}\,,\qquad V_{0\theta}(r)\equiv \frac{\partial V}{\partial\theta}\Big|_{\theta(r)=\theta_0(r)}\,. \tag{B.26}$$

Using

$$\partial_r\theta_0 = -\frac{1}{V_0}\,, \tag{B.27}$$

we obtain the following quadratic lagrangian

$$\mathcal{L} = -\frac{e^{2\phi}}{2}\,\sin\theta_0\,(e^{2h}+W_0^2)\Bigg[\frac{1}{e^{2h}+V_0^2+W_0^2}\left(V_0^3\,(\partial_r\chi)^2 - 2V_0\,V_{0\theta}\,\chi\partial_r\chi + \frac{V_{0\theta}^2}{V_0}\,\chi^2\right) + \\ +\,V_0\,(\partial_x\chi)^2\Bigg]\,. \tag{B.28}$$

Keeping the leading terms for large r, the lagrangian becomes:

$$\mathcal{L} = -\frac{r}{2}e^{2\phi}\left[\frac{n+1}{2}\,(\partial_r\chi)^2 + 2\chi\partial_r\chi + \frac{2}{n+1}\,\chi^2 + \frac{n+1}{2}\,(\partial_x\chi)^2\right], \tag{B.29}$$

and, if we represent χ as in eq. (7.6) with $l=0$, the equation of motion for ξ becomes:

$$\partial_r^2\xi + \left(2+\frac{1}{2r}\right)\partial_r\xi + \left(M^2+\frac{4n}{(n+1)^2}+\frac{1}{n+1}\frac{1}{r}\right)\xi = 0\,. \tag{B.30}$$

Now we have the following coefficients in eq. (B.4):

$$a_0 = 2\,,\qquad b_0 = M^2+\frac{4n}{(n+1)^2}\,,\qquad a_1 = \frac{1}{2}\,,\qquad b_1 = \frac{1}{n+1}\,. \tag{B.31}$$

By plugging these values in eq. (B.8) we obtain the following result for the coefficient α of the exponential:

$$\alpha = -1 \pm \sqrt{\left(\frac{n-1}{n+1}\right)^2 - M^2}\,. \tag{B.32}$$

Let us distinguish two cases, depending on the sign inside the square root. Suppose first that $M^2 \geq (\frac{n-1}{n+1})^2$ and define

$$\tilde{M}^2 = M^2 - \left(\frac{n-1}{n+1}\right)^2 . \tag{B.33}$$

In this case the values of the exponent ρ obtained from (B.9) are:

$$\rho = -\frac{1}{4} \mp \frac{n-1}{2(n+1)\tilde{M}}\, i\, . \tag{B.34}$$

It follows that the two real asymptotic solutions are:

$$\xi(r) \sim \frac{e^{-r}}{r^{\frac{1}{4}}} \cos\left[\tilde{M}r - \frac{n-1}{2(n+1)\tilde{M}} \log r\right], \qquad \frac{e^{-r}}{r^{\frac{1}{4}}} \sin\left[\tilde{M}r - \frac{n-1}{2(n+1)\tilde{M}} \log r\right]. \tag{B.35}$$

Both solutions vanish exponentially when $r \to \infty$.

If $\tilde{M}^2 < 0$, let us define $\bar{M}^2 = -\tilde{M}^2$. In this case α is real, namely $\alpha = -1 \pm \bar{M}$. Notice that $\bar{M} < 1$ and thus $\alpha < 0$. The independent asymptotic solutions are:

$$\xi(r) \sim e^{(\bar{M}-1)r} r^{-\frac{1}{4}\left(1-\frac{n-1}{(n+1)\bar{M}}\right)}, \qquad e^{-(\bar{M}+1)r} r^{-\frac{1}{4}\left(1+\frac{n-1}{(n+1)\bar{M}}\right)}, \tag{B.36}$$

and both decrease exponentially, without oscillations, when $r \to \infty$. This non-oscillatory character of the functions in eq. (B.35) make them inadequate for the type of boundary conditions we are imposing and, therefore, we shall discard them.

Notice that, for $n = 1$, the large r asymptotic solutions (B.12) and (B.35) coincide. This is of course to be expected since the abelian and non-abelian configurations coincide in the UV. It is also interesting to compare the magnitude of the fluctuation with that of the unperturbed configuration for large r. By inspecting eq. (B.25) one readily concludes that

$$\theta_0(r) \sim e^{-\frac{2}{n+1}r}, \qquad (r \to \infty)\, . \tag{B.37}$$

By comparing this behaviour with eq. (B.35) one finds

$$\frac{\xi(r)}{\theta_0(r)} \sim \frac{e^{-\frac{n-1}{n+1}r}}{r^{\frac{1}{4}}}, \qquad (r \to \infty)\, . \tag{B.38}$$

Thus, for $n \geq 1$ one has that $\xi(r)/\theta_0(r) \to 0$ as $r \to \infty$. On the contrary for $n = 0$, both in the abelian and non-abelian case, the ratio $\xi(r)/\theta_0(r)$ diverges in the UV and the first order expansion breaks down.

Let us now consider the IR behaviour of the fluctuations. Near r_* one has to leading order that $\sin\theta_0 \approx 1$ and

$$\begin{aligned} \cos\theta_0 &\approx \frac{2}{\sqrt{1+n^2}}\sqrt{r-r_*}\,, \qquad & V_0 &\approx \sqrt{n^2+1}\sqrt{r-r_*}\,, \\ W_0 &\approx \frac{n}{2}\,, & V_{0\theta} &\approx -\frac{n^2+1}{2}\,. \end{aligned} \tag{B.39}$$

JHEP12(2003)024

The IR lagrangian is of the same form as in eq. (B.14) (with $E = 0$ since we are now considering the case in which χ is independent of φ). The functions $A(r)$ to $D(r)$ are now of the form

$$\begin{aligned}
A(r) &= \Big[(n^2+1)(r-r_*)\Big]^{\frac{3}{2}}(1+o(r-r_*))\,,\\
B(r) &= \frac{(n^2+1)^{\frac{3}{2}}}{2}\sqrt{r-r_*}\,(1+o(r-r_*))\,,\\
C(r) &= \frac{1}{4}\frac{(n^2+1)^{\frac{3}{2}}}{\sqrt{r-r_*}}\,(1+o(r-r_*))\,,\\
D(r) &= \left(r_*+\frac{n^2-1}{4}\right)\sqrt{n^2+1}\,\sqrt{r-r_*}\;(1+o(r-r_*))\,.
\end{aligned} \tag{B.40}$$

Notice that, also in this case, the coefficients of the functions above are such that, after a partial integration, the singular term of $C(r)$ cancels against the leading term of $B(r)$. The differential equation that follows for ξ in the IR is:

$$\partial_r^2\xi + \left(\frac{3}{2}\frac{1}{r-r_*} + o(r-r_*)\right)\partial_r\xi + o\!\left(\frac{1}{r-r_*}\right)\xi = 0\,, \tag{B.41}$$

and, therefore, the indicial equation is the same as for eq. (B.17) (i.e. eq. (B.21)). It follows that also in this case there exists an independent solution which does not diverge when $r \to r_*$.

References

[1] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[2] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[3] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[4] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys.* **A 16** (2001) 822 [hep-th/0007018].

[5] M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9511222]

[6] J.M. Maldacena and C. Núñez, *Towards the large-N limit of pure $N = 1$ super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001].

[7] A.H. Chamseddine and M.S. Volkov, *Non-abelian BPS monopoles in $N = 4$ gauged supergravity*, *Phys. Rev. Lett.* **79** (1997) 3343 [hep-th/9707176]; *Non-abelian solitons in $N = 4$ gauged supergravity and leading order string theory*, *Phys. Rev.* **D 57** (1998) 6242 [hep-th/9711181].

[8] R. Apreda, F. Bigazzi, A.L. Cotrone, M. Petrini and A. Zaffaroni, *Some comments on $N = 1$ gauge theories from wrapped branes*, *Phys. Lett.* **B 536** (2002) 161 [hep-th/0112236].

[9] P. Di Vecchia, A. Lerda and P. Merlatti, *$N = 1$ and $N = 2$ super Yang-Mills theories from wrapped branes*, *Nucl. Phys.* **B 646** (2002) 43 [hep-th/0205204].

[10] M. Bertolini and P. Merlatti, *A note on the dual of $N = 1$ super Yang-Mills theory*, *Phys. Lett.* **B 556** (2003) 80 [hep-th/0211142].

[11] P. Merlatti, *$N = 1$ super Yang-Mills theories and wrapped branes*, *Class. Quant. Grav.* **20** (2003) S541–S546 [hep-th/0212203].

[12] G. Papadopoulos and A.A. Tseytlin, *Complex geometry of conifolds and 5-brane wrapped on 2-sphere*, *Class. and Quant. Grav.* **18** (2001) 1333 [hep-th/0012034];
A. Loewy and J. Sonnenschein, *On the holographic duals of $N = 1$ gauge dynamics*, *J. High Energy Phys.* **08** (2001) 007 [hep-th/0103163];
S.S. Gubser, A.A. Tseytlin and M.S. Volkov, *Non-abelian 4-d black holes, wrapped 5-branes and their dual descriptions*, *J. High Energy Phys.* **09** (2001) 017 [hep-th/0108205];
O. Aharony, E. Schreiber and J. Sonnenschein, *Stable non-supersymmetric supergravity solutions from deformations of the Maldacena-Núñez background*, *J. High Energy Phys.* **04** (2002) 011 [hep-th/0201224];
N. Evans, M. Petrini and A. Zaffaroni, *The gravity dual of softly broken $N = 1$ super Yang-Mills*, *J. High Energy Phys.* **06** (2002) 004 [hep-th/0203203];
T. Mateos, J.M. Pons and P. Talavera, *Supergravity dual of noncommutative $N = 1$ sym*, *Nucl. Phys.* **B 651** (2003) 291 [hep-th/0209150];
X.-J. Wang and S. Hu, *Green functions of $N = 1$ sym and radial / energy-scale relation*, *Phys. Rev.* **D 67** (2003) 105012 [hep-th/0210041];
W. Muck, *Perturbative and non-perturbative aspects of pure $N = 1$ super Yang-Mills theory from wrapped branes*, *J. High Energy Phys.* **02** (2003) 013 [hep-th/0301171];
R. Apreda, *Non supersymmetric regular solutions from wrapped and fractional branes*, hep-th/0301118.

[13] M. Bertolini, *Four lectures on the gauge-gravity correspondence*, hep-th/0303160;
F. Bigazzi, A.L. Cotrone, M. Petrini and A. Zaffaroni, *Supergravity duals of supersymmetric four dimensional gauge theories*, *Riv. Nuovo Cim.* **25N12** (2002) 1–70 [hep-th/0303191].

[14] M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, *The dirichlet super-p-branes in ten-dimensional type-IIA and IIB supergravity*, *Nucl. Phys.* **B 490** (1997) 179 [hep-th/9611159];
E. Bergshoeff and P.K. Townsend, *Super D-branes*, *Nucl. Phys.* **B 490** (1997) 145 [hep-th/9611173];
M. Aganagic, C. Popescu and J.H. Schwarz, *D-brane actions with local kappa symmetry*, *Phys. Lett.* **B 393** (1997) 311 [hep-th/9610249]; *Gauge-invariant and gauge-fixed D-brane actions*, *Nucl. Phys.* **B 495** (1997) 99 [hep-th/9612080].

[15] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158];
E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, *kappa-symmetry, supersymmetry and intersecting branes*, *Nucl. Phys.* **B 502** (1997) 149 [hep-th/9705040];
E. Bergshoeff and P.K. Townsend, *Solitons on the supermembrane*, *J. High Energy Phys.* **05** (1999) 021 [hep-th/9904020].

[16] J.D. Edelstein, A. Paredes and A.V. Ramallo, *Let's twist again: general metrics of G_2 holonomy from gauged supergravity*, *J. High Energy Phys.* **01** (2003) 011 [hep-th/0211203]; *Singularity resolution in gauged supergravity and conifold unification*, *Phys. Lett.* **B 554** (2003) 197 [hep-th/0212139].

[17] J.D. Edelstein and C. Núñez, *D6 branes and M-theory geometrical transitions from gauged supergravity*, *J. High Energy Phys.* **04** (2001) 028 [hep-th/0103167];

R. Hernandez and K. Sfetsos, *An eight-dimensional approach to G_2 manifolds*, *Phys. Lett.* **B 536** (2002) 294 [hep-th/0202135].

[18] J.P. Gauntlett, J. Gomis and P.K. Townsend, *BPS bounds for worldvolume branes*, *J. High Energy Phys.* **01** (1998) 003 [hep-th/9711205].

[19] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *J. High Energy Phys.* **06** (2002) 043 [hep-th/0205236].

[20] A. Karch, E. Katz and N. Weiner, *Hadron masses and screening from AdS wilson loops*, *Phys. Rev. Lett.* **90** (2003) 091601 [hep-th/0211107].

[21] M. Kruczenski, D. Mateos, R.C. Myers and D.J. Winters, *Meson spectroscopy in AdS/CFT with flavour*, *J. High Energy Phys.* **07** (2003) 049 [hep-th/0304032].

[22] T. Sakai and J. Sonnenschein, *Probing flavored mesons of confining gauge theories by supergravity*, *J. High Energy Phys.* **09** (2003) 047 [hep-th/0305049].

[23] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].

[24] J. Babington, J. Erdmenger, N. Evans, Z. Guralnik and I. Kirsch, *Chiral symmetry breaking and pions in non-supersymmetric gauge/gravity duals*, hep-th/0306018.

[25] M. Kruczenski, D. Mateos, R.C. Myers and D.J. Winters, *Towards a holographic dual of large-N_c QCD*, hep-th/0311270.

[26] E. Imeroni and A. Lerda, *Non-perturbative gauge superpotentials from supergravity*, hep-th/0310157.

[27] P. Ouyang, *Holomorphic D7-branes and flavored $N = 1$ gauge theories*, hep-th/0311084.

[28] X.-J. Wang and S. Hu, *Intersecting branes and adding flavors to the maldacena- Núñez background*, *J. High Energy Phys.* **09** (2003) 017 [hep-th/0307218].

[29] I. Affleck, M. Dine and N. Seiberg, *Dynamical supersymmetry breaking in supersymmetric QCD*, *Nucl. Phys.* **B 241** (1984) 493.

[30] L. Ametller, J.M. Pons and P. Talavera, *On the consistency of the $N = 1$ sym spectra from wrapped five-branes*, *Nucl. Phys.* **B 674** (2003) 231 [hep-th/0305075].

[31] M. Schvellinger and T.A. Tran, *Supergravity duals of gauge field theories from* $\mathrm{SU}(2) \times \mathrm{U}(1)$ *gauged supergravity in five dimensions*, *J. High Energy Phys.* **06** (2001) 025 [hep-th/0105019].

[32] J.M. Maldacena and H. Nastase, *The supergravity dual of a theory with dynamical supersymmetry breaking*, *J. High Energy Phys.* **09** (2001) 024 [hep-th/0105049].

[33] A.H. Chamseddine and M.S. Volkov, *Non-abelian vacua in $D = 5$, $N = 4$ gauged supergravity*, *J. High Energy Phys.* **04** (2001) 023 [hep-th/0101202].

[34] I.R. Klebanov, P. Ouyang and E. Witten, *A gravity dual of the chiral anomaly*, *Phys. Rev.* **D 65** (2002) 105007 [hep-th/0202056].

[35] U. Gursoy, S.A. Hartnoll and R. Portugues, *The chiral anomaly from M-theory*, hep-th/0311088.

JHEP12(2003)024

[36] M. Krasnitz, *A two point function in a cascading $N = 1$ gauge theory from supergravity*, hep-th/0011179;
E. Caceres and R. Hernandez, *Glueball masses for the deformed conifold theory*, *Phys. Lett.* **B 504** (2001) 64 [hep-th/0011204].

[37] J. Gomis, *On SUSY breaking and χSB from string duals*, *Nucl. Phys.* **B 624** (2002) 181 [hep-th/0111060].